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A Uniformly Differentiable Approximation Scheme for Delay Systems Using Splines.

by

K. Ito and F. Kappel

April 1989 LCDS/CCS # 89-8

A uniformly differentiable approximation scheme for delay systems using splines

K. ITO[1] and F. KAPPEL[2]

Center for Control Sciences, Brown University
and
Institute for Mathematics, University of Graz

Abstract. A new spline-based scheme is developed for linear retarded functional differential equations within the framework of semigroups on the Hilbert space $\mathbf{R}^n \times L^2$. The approximating semigroups inherit in a uniform way the characterization for differentiable semigroups from the solution semigroup of the delay system (e.g. among other things the logarithmic sectorial property for the spectrum). We prove convergence of the scheme in the state spaces $\mathbf{R}^n \times L^2$ and H^1. The uniform differentiability of the approximating semigroups enables us to establish error estimates including quadratic convergence for certain classes of initial data. We also apply the scheme for computing the feedback solutions to linear quadratic optimal control problems.

1. INTRODUCTION AND PRELIMINARIES

In this paper we consider the linear hereditary control system

$$(1.1)\qquad \begin{aligned} \dot{x}(t) &= \sum_{i=0}^{\ell} A_i x(t+\theta_i) + \int_{-r}^{0} A(\theta)x(t+\theta)\,d\theta + Bu(t), \quad t \geq 0, \\ x(0) &= \eta, \quad x(\theta) = \varphi(\theta) \text{ a.e. on } [-r,0), \\ y(t) &= Cx(t), \quad t \geq 0, \end{aligned}$$

where $-r = \theta_\ell < \cdots < \theta_0 = 0$, $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$ and $y(t) \in \mathbf{R}^p$. Furthermore, $A(\cdot)$ is an $n \times n$ matrix valued square integrable function on $[-r,0]$. It is well-known that for $(\eta,\varphi) \in Z = \mathbf{R}^n \times L^2(-r,0;\mathbf{R}^n)$ and $u \in L^2_{\mathrm{loc}}(0,\infty;\mathbf{R}^m)$ system (1.1) admits a unique solution $x \in L^2(-r,T;\mathbf{R}^n) \cap H^1(0,T;\mathbf{R}^n)$ for any $T > 0$. We define the operators $S(t)$, $t \geq 0$, by

$$S(t)(\eta,\varphi) = (x(t), x_t), \quad t \geq 0, \ (\eta,\varphi) \in Z,$$

where $x(t)$ is the solution of (1.1) with initial data (η,φ) and $u(t) \equiv 0$. The segment x_t is defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. The family $S(\cdot)$ is a strongly continuous semigroup on Z with infinitesimal generator $\mathcal{A}$ given by (see for instance [6] or [25])

$$\begin{aligned} \operatorname{dom}\mathcal{A} &= \{(\eta,\varphi) \in Z \mid \varphi \in H^1(-r,0;\mathbf{R}^n),\ \eta = \varphi(0)\}, \\ \mathcal{A}(\varphi(0),\varphi) &= (L\varphi, \dot{\varphi}) \quad \text{for } (\varphi(0),\varphi) \in \operatorname{dom}\mathcal{A}, \end{aligned}$$

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[2] Work done by this author was supported by AFOSR under Grant No. 84-0398 and by FWF(Austria) under Grants S3206 and P6005.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX where for continuous φ

$$L(\varphi) = \sum_{i=0}^{\ell} A_i\varphi(\theta_i) + \int_{-r}^{0} A(\theta)\varphi(\theta)\,d\theta.$$

If we define the input operator $\mathcal{B}\colon \mathbf{R}^m \to Z$ and the output operator $\mathcal{C}\colon Z \to \mathbf{R}^p$ by

$$\begin{aligned} \mathcal{B}u &= (Bu, 0), \quad u \in \mathbf{R}^m, \\ \mathcal{C}(\eta,\varphi) &= C\eta, \quad (\eta,\varphi) \in Z, \end{aligned} \tag{1.2}$$

then (1.1) is equivalent to the following abstract system in Z:

$$\begin{aligned} &\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \quad t \geq 0, \quad z(0) = (\eta,\varphi), \\ &y(t) = \mathcal{C}z(t), \quad t \geq 0. \end{aligned} \tag{1.3}$$

More precisely, a function $x\colon [0,\infty) \to \mathbf{R}^n$ is a solution of (1.1) if and only if the function $z(t) = (x(t), x_t)$, $t \geq 0$, is a mild solution of (1.3), i.e.,

$$z(t) = S(t)(\eta,\varphi) + \int_0^t S(t-s)\mathcal{B}u(s)\,ds, \quad t \geq 0. \tag{1.4}$$

We shall frequently make use of the following facts: The spectrum of $\mathcal{A}$ is only point spectrum and $\lambda \in \sigma(\mathcal{A})$ if and only if $\det \Delta(\lambda) = 0$, where (I_n *is the* $n \times n$ *identity matrix*)

$$\Delta(\lambda) = \lambda I_n - L(e^{\lambda\cdot} I_n), \quad \lambda \in \mathbf{C}.$$

The resolvent of $\mathcal{A}$ is given by ($\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$)

$$(\lambda I - \mathcal{A})^{-1}(\eta,\varphi) = (\psi(0), \psi), \quad \lambda \in \rho(\mathcal{A}),$$

where

$$\begin{aligned} \psi(\theta) &= e^{\lambda\theta}\psi(0) + \int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\,ds, \quad -r \leq \theta \leq 0, \\ \psi(0) &= \Delta(\lambda)^{-1}\Big(\eta + L\big(\int_\cdot^0 e^{\lambda(\cdot-s)}\varphi(s)\,ds\big)\Big). \end{aligned} \tag{1.5}$$

Together with the solution semigroup $S(\cdot)$ we shall need the adjoint semigroup $S(\cdot)^*$. Its infinitesimal generator $\mathcal{A}^*$ is given by (see for instance [6, 25])

$$\begin{aligned} \operatorname{dom}\mathcal{A}^* &= \big\{(y,\psi) \in Z \mid w \in H^1(-r,0;\mathbf{R}^n) \text{ and } \psi(-r) = A_\ell^{\mathrm{T}} y\big\}, \\ \mathcal{A}^*(y,\psi) &= \big(\psi(0) + A_0^{\mathrm{T}} y, A^{\mathrm{T}}(\cdot)y - \dot{w}\big) \quad \text{for } (y,\psi) \in \operatorname{dom}\mathcal{A}^*, \end{aligned} \tag{1.6}$$

where $w = \psi + \sum_{i=1}^{\ell-1} A_i^{\mathrm{T}} y \chi_{[-r,\theta_i)}$, χ_M denoting the characteristic function of a set M.

We shall also use the state space $H^1 = H^1(-r,0;\mathbf{R}^n)$ for system (1.1). Endowed with the inner product

$$\langle \varphi,\psi\rangle_{H^1} = \langle \varphi(0),\psi(0)\rangle_{\mathbf{R}^n} + \langle \dot\varphi,\dot\psi\rangle_{L^2}$$

H^1 is isomorphic to $\operatorname{dom}\mathcal{A}$ with the graph norm. The isomorphism $\iota\colon \operatorname{dom}\mathcal{A} \to H^1$ is given by

$$\begin{aligned} \iota(\varphi(0),\varphi) &= \varphi \quad \text{for } (\varphi(0),\varphi) \in \operatorname{dom}\mathcal{A}, \\ \iota^{-1}\varphi &= (\varphi(0),\varphi) \quad \text{for } \varphi \in H^1. \end{aligned} \tag{1.7}$$

Since $S(t)$, $t \geq 0$, restricted to $\operatorname{dom}\mathcal{A}$ forms a C_0-semigroup on $\operatorname{dom}\mathcal{A}$ (with the graph norm),

$$\tilde S(t) = \iota S(t)\iota^{-1}, \quad t \geq 0,$$

defines a C_0-semigroup on H^1. In fact, $\tilde S(t)\varphi = x_t$, where $x(t)$ is the solution of (1.1) with $u \equiv 0$ and initial data $(\varphi(0),\varphi) \in \operatorname{dom}\mathcal{A}$.

If we observe that $\int_0^t S(t-s)\mathcal{B}u(s)\,ds = (x(t),x_t)$ for $t \geq 0$ and $u \in L^2_{\mathrm{loc}}(0,\infty;\mathbf{R}^m)$, where $x(t)$ is the solution of (1.1) with initial data $\eta = 0$, $\varphi \equiv 0$, then it is not difficult to see that

$$t \to \int_0^t S(t-s)\mathcal{B}u(s)\,ds$$

defines a continuous map into $\operatorname{dom}\mathcal{A}$ with

$$\Big|\iota \int_0^t S(t-s)\mathcal{B}u(s)\,ds\Big|_{H^1} \leq m(t)|u|_{L^2(0,t;\mathbf{R}^m)}$$

for $t \geq 0$, where $m\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a nondecreasing function. Hence (1.1) with $\eta = \varphi(0)$, $\varphi \in H^1$, is also well-posed in H^1. In fact,

$$x_t = \iota z(t) = \tilde S(t)\varphi + \iota \int_0^t S(t-s)\mathcal{B}u(s)\,ds, \quad t \geq 0. \tag{1.8}$$

where $z(t)$ is given by (1.4) and $x(t)$ is the solution of (1.1) with initial data $(\varphi(0),\varphi) \in \operatorname{dom}\mathcal{A}$.

Approximation of hereditary control systems by ODE-systems has some history already. In [2] the approximating systems were obtained by using the so-called averaging projections, i.e., projections onto a subspace of step-functions. The proof of convergence as in almost all of the following papers was based on a version of the Trotter-Kato theorem in semigroup theory. In order to get higher rates of convergence in [3] a scheme was developed which uses first order or cubic splines as approximating elements. If one considers minimization of a quadratic cost criterion for system (1.1) a very desirable property of an approximation scheme is that also the adjoints of the approximating semigroups converge strongly to the adjoint of the solution semigroup of the delay system [7]. The AV-scheme developed in [2] has this property whereas the spline scheme developed in [3] does not have it. Therefore in [15] a different spline scheme was presented where also strong convergence of the adjoint semigroups is guaranteed. In contrast to the AV-scheme [22] the spline scheme of [15] does not have the property that exponential stability of the delay systems guarantees uniform (with respect to the approximation parameter) exponential stability of the approximating systems. This spline scheme still has the property of uniform output stability [16] which is enough in order to deal with the infinite time horizon problem for (1.1) [17]. In [12, 13, 10] a scheme using Legendre polynomials as approximating elements was developed. The construction is based on Lanczos' tau-method. The scheme has all the qualitative properties which are stated above to be true for the AV-scheme. Also using Legendre polynomials a different scheme was developed in [14] using the basic ideas of [15]. Numerical evidence indicates that this Legendre-scheme has analogous properties as the Legendre-tau-scheme of [12], but uniform exponential stability for the approximating systems has not been established yet. Using piecewise linear functions in [21] a scheme was developed in the spirit of [15] which also has all the qualitative properties mentioned above for the AV and Legendre-tau schemes. In all the papers mentioned up to now no convergence rates have been established or only convergence rates which obviously are not optimal. For instance in [3] only linear convergence for smooth data in case of first order splines is established. In a recent paper [19] I. Lasiecka and A. Manitius gave for the first time optimal rates for the AV-scheme. These estimates are essentially based on uniform (with respect to the approximation parameter) differentiability of the approximating semigroups, which means that the characterization of differentiable semigroups in [20; Theorem 4.7] is uniformly valid for the approximating semigroups.

In this paper we develop a scheme using first order splines which essentially has all the good properties of the AV-scheme. In addition we are able to establish error estimates analogous to those in [19]. Naturally uniform differentiability is also the essential basis for our approach which as far as convergence rates and uniform exponential stability are concerned is motivated by the ideas of [19].

In Section 2 we give the basic ideas for the construction of the approximation scheme using two sequences of subspaces leading to two sequences of approximating semigroups in $\mathbf{R}^N \times L^2$ resp. $\operatorname{dom} \mathcal{A}$. We also give matrix representations for the approximating generators and the approximating input resp. output operators. Furthermore we compute the resolvent operator for the approximating generators. In Section 3 we prove convergence of the approximating semigroups in both spaces by applying a version of the Trotter-Kato-Theorem for C_0-semigroups. For the scheme in the state space Z we get also strong convergence of the adjoint semigroups. One should be aware of the simplicity of the consistency arguments given in Subsection 3.2 for the approximating semigroups and their adjoints. Note that strong convergence also for the adjoint semigroups is needed for the proof of Theorem 7.5. The results of Section 4 are fundamental for the rest of the paper. In this section we prove that the approximating semigroups are differentiable uniformly with respect to N. This allows to use the ideas of [19] for the proof of uniform exponential stability and for proving rate estimates. Uniform exponential stability is established in Section 5, whereas rate estimates are given in Section 6. Implications of the convergence results for the approximation of the linear-quadratic optimal control problem on the infinite time interval are discussed in Section 7. Finally, in Appendices A and B we collect some of the more technical estimates used in Sections 4–6, respectively.

2. Formulation of the scheme

In this section we define the spline approximation scheme and prove some basic properties. Let $t_k^N = -kr/N$, $k = 0, \dots, N$, and $t_{-1}^N = 0$, $t_{N+1}^N = -r$ for $N = 1, 2, \dots$. With B_k^N, $k = 0, \dots, N$, we denote the usual first order basis splines on the interval $[-r, 0]$ corresponding to the mesh $t_0^N, \dots, t_N^N$,

$$B_k^N(\theta) = \begin{cases} \frac{N}{r}(\theta - t_{k+1}^N) & \text{for } t_{k+1}^N \le \theta \le t_k^N, \\ \frac{N}{r}(t_{k-1}^N - \theta) & \text{for } t_k^N \le \theta \le t_{k-1}^N, \\ 0 & \text{elsewhere.} \end{cases}$$

Furthermore we put

$$E_k^N = \chi_{[t_k^N, t_{k-1}^N)}, \quad k = 1, \dots, N,$$

and

$$\hat{E}_0^N = (I_n, 0), \quad \hat{E}_k^N = (0, E_k^N I_n), \quad k = 1, \dots, N.$$

The following spaces will be used in the sequel:

$$\begin{aligned} W^N &= \operatorname{span}(E_1^N I_n, \dots, E_N^N I_n) \subset L^2(-r, 0; \mathbf{R}^n), \\ Z^N &= \mathbf{R}^n \times W^N = \operatorname{span}(\hat{E}_0^N, \dots, \hat{E}_N^N) \subset Z, \\ X^N &= \operatorname{span}(B_0^N I_n, \dots, B_N^N I_n) \subset H^1, \\ Z_1^N &= \iota^{-1} X^N \subset \operatorname{dom} \mathcal{A}. \end{aligned}$$

It is convenient to introduce the "basis matrices"

$$\begin{aligned} E^N &= (E_1^N I_n \cdots E_N^N I_n), \\ \hat{E}^N &= (\hat{E}_0^N \cdots \hat{E}_N^N), \\ B^N &= (B_0^N I_n \cdots B_N^N I_n), \\ \iota^{-1} B^N &= (\iota^{-1} B_0^N I_n \cdots \iota^{-1} B_N^N I_n). \end{aligned}$$

Any $z = (\eta, \varphi) \in Z^N$ can be written as $z = (\eta, E^N a^N) = \hat{E}^N \operatorname{col}(\eta, a_1^N, \dots, a_N^N)$, where $a^N = \operatorname{col}(a_1^N, \dots, a_N^N)$, $a_k^N \in \mathbf{R}^n$, is the coordinate vector of $\varphi \in W^N$ with respect to the basis E^N. Similarly any $\varphi \in X^N$ is given by $\varphi = B^N b^N$ with $b^N = \operatorname{col}(b_0^N, \dots, b_N^N)$, $b_k^N \in \mathbf{R}^n$.

The orthogonal projections $P^N : Z \to Z^N$ and $P_1^N : H^1 \to X^N$ are characterized in

Lemma 2.1. *a) For* $(\eta, \varphi) \in Z$

$$P^N z = (\eta, E^N a^N), \quad \textit{where} \quad a_k^N = \frac{N}{r} \int_{t_k^N}^{t_{k-1}^N} \varphi(s)\, ds, \quad k = 1, \dots, N.$$

b) For $\psi \in H^1$

$$P_1^N \psi = B^N b^N, \quad \textit{where} \quad b_k^N = \psi(t_k^N), \quad k = 0, \dots, N.$$

PROOF: The results follow by easy computations from $\langle z - P^N z, \hat{E}_k^N \rangle_Z = 0$, $k = 0, \dots, N$, and $\langle \psi - P_1^N \psi, B_k^N I_n \rangle_{H^1} = 0$, $k = 0, \dots, N$, respectively. ∎

Remark. The subspaces Z^N were used for the socalled scheme of averaging projections introduced in [2], whereas Z_1^N are the spline subspaces of [3]. Note that $\iota^{-1} P_1^N$ is not the orthogonal projection $\operatorname{dom} \mathcal{A} \to Z_1^N$ with respect to the inner product in Z.

By definition of the spaces Z_1^N, Z^N and the operators $\mathcal{A}$, $\mathcal{B}$ we have

$$
\begin{aligned}
\mathcal{A} z^N &\in Z^N \quad \text{for any } z^N \in Z_1^N, \\
\mathcal{B}\xi &\in Z^N \quad \text{for any } \xi \in \mathbf{R}^m.
\end{aligned} \tag{2.1}
$$

For a mild solution $z(t)$, $t \geq 0$, of (1.3) as given by (1.4) we seek an approximation $w^N(t) \in Z_1^N$, $t \geq 0$. If $z(t)$ is a strong (i.e., differentiable) solution of (1.3) then $\dot{z}(t)$ is in general not in $\operatorname{dom} \mathcal{A}$ but in the subspace generated by $\mathcal{A}z(t) + \mathcal{B}u(t)$. By (2.1) we have $\mathcal{A}w^N(t) + \mathcal{B}u(t) \in Z^N$, $t \geq 0$. On the other hand $\dot{w}^N(t)$ is in $Z_1^N \subset \operatorname{dom} \mathcal{A}$. The above consideration concerning strong solutions of (1.3) motivate to determine $w^N(t)$ such that

$$
P^N \dot{w}^N(t) = \frac{d}{dt} P^N w^N(t) = \mathcal{A} w^N(t) + \mathcal{B} u(t), \quad t \geq 0. \tag{2.2}
$$

Remark. If instead of (2.2) one imposes the condition

$$
\dot{w}^N(t) = P^N \big(\mathcal{A} w^N(t) + \mathcal{B} u(t) \big), \quad t \geq 0,
$$

one obtains the spline scheme of [3] which lacks a number of qualitative properties one would like to have (see the introduction).

In order to derive a differential equation for the coordinate vector of $w^N(t)$ we shall need

LEMMA 2.2. *a) P^N restricted to Z_1^N is a bijection $Z_1^N \to Z^N$. Its matrix representation (with respect to the basis $\iota^{-1} B^N$ of Z_1^N and the basis $\hat{E}^N$ of Z^N) is given by*

$$
Q^N = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
1/2 & 1/2 & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1/2 & 1/2
\end{pmatrix} \otimes I_n \in \mathbf{R}^{n(N+1) \times n(N+1)}.
$$

b) $\mathcal{A}$ restricted to Z_1^N is a map $Z_1^N \to Z^N$ with the matrix representation

$$
H^N = \begin{pmatrix}
D_0^N & \cdots & \cdots & D_N^N \\
0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots \\
\vdots & & & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix} + \frac{N}{r} \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
1 & -1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1 & -1
\end{pmatrix} \otimes I_n \in \mathbf{R}^{n(N+1) \times n(N+1)}.
$$

where $D_k^N = L(B_k^N) = \sum_{j=0}^{\ell} A_j B_k^N(\theta_j) + \int_{-r}^{0} A(\theta) B_k^N(\theta)\, d\theta$, $k = 0, \dots, N$.
c) The matrix representation of $\mathcal{B}$ considered as a map into Z^N is

$$B^N = \begin{pmatrix} B \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{R}^{n(N+1)\times m},$$

whereas the matrix representation of $\mathcal{C}$ restricted to Z^N is

$$C^N = (C \quad 0 \quad \cdots \quad 0) \in \mathbf{R}^{p\times n(N+1)}.$$

PROOF: For $z = (\varphi(0), \varphi) \in Z_1^N$ with $\varphi = B^N b^N$ we get (using Lemma 2.1.a)) $P^N z = (\eta, E^N a^N)$, where $\eta = \varphi(0) = b_0^N$ and

$$a_k^N = \frac{N}{r}\int_{t_k^N}^{t_{k-1}^N} \varphi(\theta)\, d\theta = \frac{N}{r}\sum_{i=0}^{N} b_i^N \int_{t_k^N}^{t_{k-1}^N} B_k^N(\theta)\, d\theta = \frac{1}{2}(b_{k-1}^N + b_k^N), \quad k = 1, \dots, N.$$

Part b) follows from easy computations using the definition of $\mathcal{A}$ and part c) is trivial. ∎

Let $L^N = (P^N|_{Z_1^N})^{-1}$, $\mathcal{A}^N = \mathcal{A}L^N$ and put $z^N(t) = P^N u^N(t)$, $t \geq 0$. Then (2.2) is equivalent to

$$\dot{z}^N(t) = \mathcal{A}^N z^N(t) + \mathcal{B}u(t), \quad t \geq 0. \tag{2.3}$$

$\mathcal{A}^N$ is an operator $Z^N \to Z^N$. Note that (2.3) is an ordinary differential equation on Z^N. In view of Lemma 2.2 the matrix representation A^N of $\mathcal{A}^N$ is given by

$$A^N = H^N (Q^N)^{-1}. \tag{2.4}$$

Let $b^N(t)$ and $a^N(t)$ be the coordinate vectors of $u^N(t)$ and $z^N(t)$, respectively, i.e., $u^N(t) = B^N b^N(t)$ and $z^N(t) = \hat{E}^N a^N(t)$. Then

$$a^N(t) = Q^N b^N(t), \quad t \geq 0,$$

and equations (2.2) and (2.3) are equivalent to

$$Q^N \dot{b}^N(t) = H^N b^N(t) + B^N u(t), \quad t \geq 0, \tag{2.5}$$

and

$$\dot{a}^N(t) = A^N a^N(t) + B^N u(t), \quad t \geq 0, \tag{2.6}$$

respectively.

Remarks. 1. If $\mathcal{A}^{-1}$ exists, then $\mathcal{A}Z_1^N = Z^N$. Hence, the approximation scheme given in this paper is equivalent to the scheme developed in [9] for the approximation of the spectrum of $\mathcal{A}$ using first order splines.
2. The approximation scheme developed here can be regarded as a spline-tau approximation (in the sense of Lanczos' so-called tau-method). In [12, 13, 10] the Legendre-tau approximation for delay systems was developed using Legendre-polynomials instead of splines for the definition of the spaces Z^N and X^N. The Legendre-tau scheme was formulated in a similar manner as above (see equations (2.3) and (2.4) in [10]).
3. Using different spline elements for the definition of the subspaces Z^N and X^N one can get a whole family of approximation schemes using the ideas presented in this paper. We conjecture that some of them lead to approximations of the characteristic equation of the delay systems by functions which involve the Padé approximations of the exponential function in the main diagonal of the Padé table (see the remarks before Lemma 2.4)

For the proof of convergence for the scheme presented in this section we shall need an explicit representation for the resolvent operators $(\lambda I - \mathcal{A}^N)^{-1}$. We introduce the function

$$\mu(\tau) = \frac{1-\tau/2}{1+\tau/2}, \quad \tau \in \mathbf{C},\ \tau \neq 2,$$

which is the first non-constant entry in the main diagonal of the Padé table for $e^{-\tau}$ (cf. [1]). Next we define

$$e_\lambda^N(\theta) = \sum_{k=0}^{N} B_k^N(\theta)\mu(r\lambda/N)^k, \quad -r \le \theta \le 0,\ \lambda \neq -2N/r, \tag{2.7}$$

which is an approximation to $e^{\lambda\theta}$ on $[-r, 0]$ and put

$$\Delta^N(\lambda) = \lambda I_n - L(e_\lambda^N I_n), \quad \lambda \neq -2N/r.$$

PROPOSITION 2.3. *a) A complex number $\lambda \neq -2N/r$ is in $\sigma(\mathcal{A}^N)$ if and only if*

$$\det \Delta^N(\lambda) = 0.$$

b) Let $\lambda \notin \sigma(\mathcal{A}^N)$, $\lambda \neq -2N/r$ and $z = (\eta, E^N a^N) \in Z^N$. Then

$$(\lambda I - \mathcal{A}^N)^{-1} z = P^N(\psi(0), \psi),$$

where

$$\psi(0) = \Delta^N(\lambda)^{-1}(\eta + L(\tau^N)), \qquad \psi = e_\lambda^N \psi(0) + \tau^N$$

with

$$\tau^N(\theta) = \frac{1}{2}\frac{r}{N}(1+\mu(r\lambda/N))\sum_{k=1}^{N} B_k^N(\theta)\sum_{j=1}^{k}\mu(r\lambda/N)^{k-j}a_j^N$$

$$= \frac{r}{N}\sum_{k=1}^{N} B_k^N(\theta)\sum_{j=1}^{k}\frac{1}{2}\Big(\mu(r\lambda/N)^{k-j} + \mu(r\lambda/N)^{k-j+1}\Big)a_j^N.$$

PROOF: Given $z \in Z^N$ we solve the equation $(\lambda I - \mathcal{A}^N)w = z$, $w = \hat{E}^N c^N \in Z^N$. This equation is equivalent to (see (2.4))

$$(\lambda Q^N - H^N)(Q^N)^{-1}c^N = a^N. \tag{2.8}$$

We put $b^N = (Q^N)^{-1}c^N$, i.e., $w = P^N \iota^{-1}\psi$ with $\psi = B^N b^N$. The definitions of Q^N and H^N (see Lemma 2.2) immediately show that (2.8) is equivalent to

$$\lambda b_0^N - \sum_{k=0}^{N} D_k^N b_k^N = a_0^N, \tag{2.9}$$

$$(1 + \frac{r\lambda}{2N})b_k^N = (1 - \frac{r\lambda}{2N})b_{k-1}^N + \frac{r}{N}a_k^N, \quad k = 1, \dots, N. \tag{2.10}$$

If $\lambda \neq -2N/r$ then (2.10) is equivalent to

$$\begin{aligned} b_k^N &= \mu(r\lambda/N)^k b_0^N + \frac{r}{2N}(1 + \mu(r\lambda/N)) \sum_{j=1}^{k} \mu(r\lambda/N)^{k-j} a_j^N \\ &= \epsilon_\lambda^N(t_k^N)b_0^N + \tau^N(t_k^N), \quad k = 1, \dots, N. \end{aligned} \tag{2.11}$$

Therefore in case $\lambda \neq -2N/r$ equations (2.9) and (2.10) are equivalent to (2.11) and

$$\begin{aligned} \Delta^N(\lambda)b_0^N &= \eta + \frac{r}{2N}(1 + \mu(r\lambda/N)) \sum_{k=1}^{N} D_k^N \sum_{j=1}^{k} \mu(r\lambda/N)^{k-j} a_j^N \\ &= \eta + L(\tau^N). \end{aligned}$$

This proves that $\lambda \neq -2N/r$ is in $\sigma(\mathcal{A}^N)$ if and only if $\det \Delta^N(\lambda) = 0$. Moreover, we immediately see that for $\lambda \notin \sigma(\mathcal{A}^N)$, $\lambda \neq -2N/r$, the element $(\lambda I - \mathcal{A}^N)^{-1}z$ is given as stated in the proposition. ∎

Remarks. 1. Using the definition of ϵ_λ^N we see that

$$\tau^N(t_k^N) = \int_{t_k^N}^{0} \epsilon_\lambda^N(t_k^N - s)E^N(s)a^N\, ds, \tag{2.12}$$

i.e., τ^N is the interpolating spline for the function $\theta \to \int_\theta^0 \epsilon_\lambda^N(\theta - s)E^N(s)a^N\, ds$.

2. In case $\lambda = -2N/r$ equations (2.10) are equivalent to

$$b_k^N = -\frac{r}{N}(1 - \frac{r\lambda}{2N})^{-1} a_{k+1}^N, \quad k = 0, \dots, N-1.$$

This together with (2.9) gives

$$D_N^N b_N^N = -a_0^N - \frac{r}{N}(1 - \frac{r\lambda}{2N})^{-1}\left(\frac{2N}{r}a_1^N - \sum_{k=0}^{N-1} D_k^N a_{k+1}^N\right).$$

Therefore $\lambda = -2N/r$ is in $\sigma(\mathcal{A}^N)$ if and only if $\det D_N^N = 0$. In case $A(\cdot) \equiv 0$ and N sufficiently large this is equivalent to $\det A_\ell = 0$, a condition which is equivalent to the existence of so-called small solutions for the uncontrolled delay system (see [8]).
3. In the single delay case $L(\varphi) = A_0\varphi(0) + A_1\varphi(-r)$ we have

$$\Delta^N(\lambda) = \lambda I_n - A_0 - \left(\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right)^N A_1, \quad \lambda \neq -2N/r.$$

For the other schemes discussed in the introduction $\Delta^N(\lambda)$ in this simple case is given by $\Delta^N(\lambda) = \lambda I_n - A_0 - f^N(\lambda)A_1$, where $f^N(\lambda)$ is a rational approximation for $e^{-\lambda r}$. For the scheme of averaging projections we have $f^N(\lambda) = (1 + \lambda r/N)^{-N}$ (see [2]) for the Legendre-tau scheme of [12] $f^N(\lambda)$ is the N-th entry in the main diagonal of the Padé table for $e^{-\lambda r}$, whereas for the Legendre-scheme of [14] $f^N(\lambda)$ is the proper rational entry at position $(N, N-1)$ in the Padé table for $e^{-\lambda r}$. For the piecewise linear scheme of [21] we have $f^N(\lambda) = \left(\frac{6-2r\lambda/N}{6+4r\lambda/N+(r\lambda/N)^2}\right)^N$ (note that $\frac{6-2r}{6+4r+r^2}$ is a Padé approximant for e^{-r}). In case of the scheme developed in [15] $f^N(\lambda)$ cannot be given explicitely but is obtained by a recursion formula.

For later use we state a simple observation:

LEMMA 2.4. *Let* $\gamma = \sum_{i=0}^{\ell} |A_i| + \int_{-r}^{0} |A(\theta)|\, d\theta$. *Then* $\det \Delta^N(\lambda) \neq 0$ *for all* $\lambda \in \mathbf{C}$ *with* $\operatorname{Re}\lambda \geq 0$, $|\lambda| > \gamma$, $N = 1, \dots$.

PROOF: Obviously $|(1 - r\lambda/2N)(1 + r\lambda/2N)^{-1}| \leq 1$ for $\operatorname{Re}\lambda \geq 0$. Therefore $|\epsilon_\lambda^N(\theta)| \leq 1$ for $\operatorname{Re}\lambda \geq 0$ and $-r \leq \theta \leq 0$ (see (2.7) for the definition of $\epsilon_\lambda^N(\theta)$). Thus

$$\left|\sum_{i=0}^{\ell} A_i \epsilon_\lambda^N(\theta_i) + \int_{-r}^{0} A(\theta)\epsilon_\lambda^N(\theta)\, d\theta\right| \leq \gamma$$

and the result follows from

$$\Delta^N(\lambda) = \lambda\Big(I_n - \frac{1}{\lambda}\Big(\sum_{i=0}^{\ell} A_i \epsilon_\lambda^N(\theta_i) + \int_{-r}^{0} A(\theta)\epsilon_\lambda^N(\theta)\, d\theta\Big)\Big). \quad \blacksquare$$

3. Convergence of the scheme

In this section we establish convergence of the approximation scheme presented in the preceding section for both state spaces, Z and H^1. Moreover, we also prove strong convergence of the adjoint semigroups $S^N(t)^*$ to $S(t)^*$ in the state space Z. We first consider convergence in Z. For that purpose a version of the Trotter-Kato theorem as given in [18] will be used.

Let $(Y^N, \|\cdot\|_N)$ be a sequence of Banach spaces which converges in the sense of Kato to a Banach space $(Y, \|\cdot\|)$ (see [18, Ch.IX, §3]), i.e., for each N there exists a bounded linear operator $P^N: Y \to Y^N$ such that

(K 1) $\|P^N\| \le c_1$ for all N with c_1 independent of N,

(K 2) $\lim_{N\to\infty} \|P^N y\|_N = \|y\|$ for all $y \in Y$,

(K 3) there exists a constant $c_2 > 0$ such that for all N and all $y \in Y^N$ there exists an $x \in Y$ with
$$y = P^N x \quad \text{and} \quad \|x\| \le c_2 \|y\|_N.$$

Under these conditions the following result is valid:

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{A}^N$ be the infinitesimal generators of strongly continuous semigroups $S(t)$ on Y and $S^N(t)$ on Y^N, respectively. Suppose that*

(i) *for some constants $\omega \in \mathbf{R}$ and $M \ge 1$*
$$\|S^N(t)\|_N \le M e^{\omega t} \quad \text{for all } t \ge 0 \text{ and all } N$$

and

(ii) *for some $\lambda \in \rho(\mathcal{A}) \cap \bigcap_{N=1}^{\infty} \rho(\mathcal{A}^N)$ and all $x \in Y$*
$$\lim_{N\to\infty} \|(\lambda I - \mathcal{A}^N)^{-1} P^N x - P^N(\lambda I - \mathcal{A})^{-1} x\|_N = 0.$$

Then for all $x \in Y$
$$\lim_{N\to\infty} \|S^N(t) P^N x - P^N S(t) x\|_N = 0$$
uniformly for t in bounded intervals.

Proof: The proof of this theorem uses exactly the same arguments as those given in [18, Ch.IX, Theorem 2.16] and is based on the equality

$$\begin{aligned}(3.1)\qquad &(\lambda I - \mathcal{A}^N)^{-1}(S^N(t)P^N - P^N S(t))(\lambda I - \mathcal{A})^{-1} \\ &\quad = \int_0^t S^N(t-s)\Big((\lambda I - \mathcal{A}^N)^{-1}P^N - P^N(\lambda I - \mathcal{A})^{-1}\Big)S(s)\,ds.\end{aligned}$$

Then for any $y \in \operatorname{dom} \mathcal{A}$

$$(3.2)\qquad \lim_{N\to\infty} \|(\lambda I - \mathcal{A}^N)^{-1}(S^N(t)P^N - P^N S(t))y\|_N = 0$$

uniformly for t in bounded intervals, where we have used assumptions (i) and (ii) together with Lebesgue's dominated convergence theorem. Next we consider, for $x \in Y$,

$$
(3.3)\qquad
\begin{aligned}
(P^N S(t) - S^N(t)P^N)(\lambda I - \mathcal{A})^{-1}x \\
&= \Big(P^N(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{A}^N)^{-1}P^N\Big)S(t)x \\
&+ (\lambda I - \mathcal{A}^N)^{-1}\Big(P^N S(t) - S^N(t)P^N\Big)x \\
&+ S^N(t)\Big((\lambda I - \mathcal{A}^N)^{-1}P^N - P^N(\lambda I - \mathcal{A})^{-1}\Big)x.
\end{aligned}
$$

The second and third term on the right-hand side tend to zero uniformly on bounded t-intervals as $N \to \infty$, because of assumptions (i), (ii) and (3.2). The same is true also for the first term on the right-hand side of (3.3), because for any $T > 0$ the set $\{S(t)x \mid 0 \le t \le T\}$ is compact. This shows that

$$
\lim_{N\to\infty} (P^N S(t) - S^N(t)P^N)y = 0
$$

uniformly on bounded t-intervals for any $y \in \operatorname{dom}\mathcal{A}^2$. A density argument finishes the proof. ∎

In our situation $Y^N = Z^N$ and P^N is the orthogonal projection $Z \to Z^N$. We have to establish hypotheses (K 1)–(K 3) and assumptions (i), (ii) of Theorem 3.1.

3.1. Uniform dissipativity. Following an idea which goes back to [26] we shall use equivalent norms $\|\cdot\|_N$ on Z in order to establish the dissipativity properties which imply assumption (i) of Theorem 3.1.

For $j = 1, \ldots, \ell - 1$ define the indices k_j^N by $\theta_j \in [t^N_{k_j^N}, t^N_{k_j^N - 1})$. If N is sufficiently large (which we assume from now on) then

$$
1 < k_1^N < \cdots < k_{\ell-1}^N < N.
$$

The step functions $g^N(\theta) = \sum_{k=1}^N g_k^N \chi_{[t_k^N, t_{k-1}^N)}$ are defined by

$$
(3.4)\qquad
\begin{aligned}
g_N^N &= 1 + \frac{r}{N}, \\
g_k^N &= g_{k+1}^N + \frac{r}{N} \quad \text{for } k = 1, \ldots, N-1 \text{ with } k \notin \{k_j^N \mid j = 1, \ldots, \ell - 1\}, \\
g_k^N &= g_{k+1}^N + \frac{r}{N} + B_k^N(\theta_j) \quad \text{for } k = k_j \text{ and } k = k_j - 1,\ j = 1, \ldots, \ell - 1.
\end{aligned}
$$

It is obvious that

$$
g(\theta) := \lim_{N\to\infty} g^N(\theta) = r + \theta + \sum_{j=1}^{\ell} \chi_{[\theta_j, 0)} \quad \text{a.e. on } [-r, 0].
$$

The inner products $\langle\cdot,\cdot\rangle_N$ and $\langle\cdot,\cdot\rangle_g$ are defined by

$$\langle(\eta,\varphi),(\rho,\psi)\rangle_N = \eta^{\mathrm{T}}\rho + \int_{-r}^{0}\varphi(\theta)^{\mathrm{T}}\psi(\theta)g^N(\theta)\,d\theta,$$
$$\langle(\eta,\varphi),(\rho,\psi)\rangle_g = \eta^{\mathrm{T}}\rho + \int_{-r}^{0}\varphi(\theta)^{\mathrm{T}}\psi(\theta)g(\theta)\,d\theta,$$

for (η,φ), $(\rho,\psi)\in Z$. Because of $1\le g(\theta), g^N(\theta)\le \ell + r$, $-r\le\theta\le 0$, the corresponding norms $\|\cdot\|_N$, $\|\cdot\|_g$ and $\|\cdot\|$ on Z are equivalent uniformly with respect to N,

$$\|z\|\le\|z\|_N,\quad \|z\|_g\le(\ell+r)^{1/2}\|z\|\quad \text{for } z\in Z.$$

Therefore hypotheses (K 1) and (K 3) are satisfied for $(Z^N,\|\cdot\|_N)$ and $(Z,\|\cdot\|_g)$. For $z=(\eta,\varphi)$ and $P^Nz=(\eta,\varphi^N)$ we get

$$\begin{aligned}\big|\|P^Nz\|_N^2-\|z\|_g^2\big| &= \big|\|(g^N)^{1/2}\varphi^N\|_{L^2}^2-\|g^{1/2}\varphi\|_{L^2}^2\big|\\ &\le \|(g^N)^{1/2}\varphi^N-g^{1/2}\varphi\|_{L^2}\left(\|(g^N)^{1/2}\varphi^N\|_{L^2}+\|g^{1/2}\varphi\|_{L^2}\right)\\ &\le \Big((\ell+r)^{1/2}\|\varphi^N-\varphi\|_{L^2}+\|(g^N-g)^{1/2}\varphi\|_{L^2}\Big)\Big(\|(g^N)^{1/2}\varphi^N\|_{L^2}+\|g^{1/2}\varphi\|_{L^2}\Big).\end{aligned}$$

This estimate shows that $\lim_{N\to\infty}\|P^Nz\|_N=\|z\|_g$ for all $z\in Z$, i.e., assumption (K 2) is also satisfied.

LEMMA 3.2. *Let $S^N(t)$, $t\ge 0$, be the semigroup on Z^N generated by $\mathcal{A}^N$.*
a) For all $z\in Z^N$, $z=(\eta,\varphi)$,

$$\langle\mathcal{A}^Nz,z\rangle_N\le\omega_N|\eta|^2-\frac{1}{4}\|\varphi\|_{L^2}^2$$

for N sufficiently large, where $\omega_N=\omega+r/N$ with

$$\omega=\frac{1}{2}\lambda_{\max}(A_0+A_0^{\mathrm{T}})+\frac{1}{2}(\ell+r)+\frac{1}{2}\sum_{j=1}^{\ell}|A_j|^2+\|A\|_{L^2}^2.$$

b) For N sufficiently large

$$\|S^N(t)\|_N\le e^{\omega_N t}\quad\text{and}\quad\|S^N(t)^*\|_N\le e^{\omega_N t},\quad t\ge 0.$$

PROOF: Part b) is an immediate consequence of part a), because $\langle\mathcal{A}^Nz,z\rangle_N\le\omega_N\|z\|_N^2$ for all $z\in Z^N$. In order to prove a) let $z=\hat{E}_Na^N\in Z^N$ and $L^Nz=(\varphi^N(0),\varphi^N)$ with $\varphi^N=B^Nb^N$. Then

$$a^N=Q^Nb^N. \tag{3.5}$$

For elements $z_1 = \hat{E}^N c^N$ and $z_2 = \hat{E}^N d^N$ a simple computation yields

$$\langle z_1, z_2\rangle_N = (c^N)^{\mathrm{T}} R^N d^N, \tag{3.6}$$

where $R^N = \operatorname{diag}\left(1, \frac{r}{N} g_1^N, \dots, \frac{r}{N} g_N^N\right) \otimes I_n$. Using (3.5), (3.4) and (2.4) we get

$$\langle \mathcal{A}^N z, z\rangle_N = (b^N)^{\mathrm{T}} (H^N)^{\mathrm{T}} R^N Q^N b^N.$$

Simple computations using (3.4) and the matrices H^N, Q^N as given in Lemma 2.2 show that

$$\begin{aligned}
\langle \mathcal{A}^N z, z\rangle_N &= \frac{1}{2} g_1^N |b_0^N|^2 + \frac{1}{2}\sum_{k=1}^{N-1} (g_{k+1}^N - g_k^N)|b_k^N|^2 - \frac{1}{2} g_N^N |b_N^N|^2 \\
&\quad + (b_0^N)^{\mathrm{T}} D_0^N b_0^N + (b_0^N)^{\mathrm{T}} \sum_{k=1}^{N} D_k^N b_k^N \\
&= \frac{1}{2}(\ell + r)|b_0^N|^2 - \frac{r}{2N}\sum_{k=1}^{N} |b_k^N|^2 - \frac{1}{2}|b_N^N|^2 \\
&\quad - \frac{1}{2}\sum_{j=1}^{\ell-1}\left(B_{k_j^N}^N(\theta_j)|b_{k_j^N}^N|^2 + B_{k_j^N-1}^N(\theta_j)|b_{k_j^N-1}^N|^2\right) + (b_0^N)^{\mathrm{T}} A_0 b_0^N \\
&\quad + (b_0^N)^{\mathrm{T}} \sum_{k=1}^{N}\left(\sum_{j=1}^{\ell} A_j B_k^N(\theta_j)\right) b_k^N + (b_0^N)^{\mathrm{T}} \int_{-r}^{0} A(\theta)\left(\sum_{k=0}^{N} B_k^N(\theta) b_k^N\right) d\theta.
\end{aligned} \tag{3.7}$$

Observing $B_k^N(\theta_j) = 0$ for $k \neq k_j^N$ and $k \neq k_j^N - 1$ we get

$$\begin{aligned}
&\left|(b_0^N)^{\mathrm{T}} \sum_{k=1}^{N}\left(\sum_{j=1}^{\ell} A_j B_k^N(\theta_j)\right) b_k^N\right| \\
&\quad \le |b_0^N| \sum_{j=1}^{\ell} |A_j|\left|B_{k_j^N}^N(\theta_j) b_{k_j^N}^N + B_{k_j^N-1}^N(\theta_j) b_{k_j^N-1}^N\right| + |b_0^N|\,|A_\ell|\,|b_N^N| \\
&\quad \le \frac{1}{2}|b_0^N|^2 \sum_{j=1}^{\ell} |A_j|^2 + \frac{1}{2}\sum_{j=1}^{\ell-1}\left|B_{k_j^N}^N(\theta_j) b_{k_j^N}^N + B_{k_j^N-1}(\theta_j) b_{k_j^N-1}^N\right|^2 + \frac{1}{2}|b_N^N|^2.
\end{aligned}$$

Using $\lambda\alpha^2 + (1-\lambda)\beta^2 - \left(\lambda\alpha + (1-\lambda)\beta\right)^2 \ge 0$ for $0 \le \lambda \le 1$ and $\alpha, \beta \ge 0$ we get

$$\begin{aligned}
&\left|(b_0^N)^{\mathrm{T}} \sum_{k=1}^{N}\left(\sum_{j=1}^{\ell} A_j B_k^N(\theta_j)\right) b_k^N\right| \\
&\quad \le \frac{1}{2}|b_0^N|^2 \sum_{j=1}^{\ell} |A_j|^2 + \frac{1}{2}|b_N^N|^2 \\
&\quad\quad + \frac{1}{2}\sum_{j=1}^{\ell-1}\left(B_{k_j^N}^N(\theta_j)|b_{k_j^N}^N|^2 + B_{k_j^N-1}^N(\theta_j)|b_{k_j^N-1}^N|^2\right).
\end{aligned} \tag{3.8}$$

Equation (3.5) implies

$$(3.9) \qquad \|\varphi\|^2_{L^2} = \frac{r}{N}\sum_{k=1}^{N}|a_k^N|^2 = \frac{r}{4N}\sum_{k=1}^{N}|b_{k-1}^N + b_k^N|^2 \le \frac{r}{2N}\sum_{k=1}^{N}\left(|b_{k-1}^N|^2 + |b_k^N|^2\right).$$

From $\varphi^N(\theta) = \frac{N}{r}\left((t_{k-1}^N - \theta)b_k^N + (\theta - t_{k-1}^N)b_{k-1}^N\right)$, $t_k^N \le \theta \le t_{k-1}^N$, we obtain

$$\|\varphi^N\|^2_{L^2} = \frac{r}{3N}\sum_{k=1}^{N}\Big(|b_k^N|^2 + (b_k^N)^{\mathrm{T}}b_{k-1}^N + |b_{k-1}^N|^2\Big) \le \frac{r}{2N}\sum_{k=1}^{N}\left(|b_{k-1}^N|^2 + |b_k^N|^2\right).$$

This implies

$$(3.10) \qquad \begin{aligned} &\left|(b_0^N)^{\mathrm{T}}\int_{-r}^{0}A(\theta)\Big(\sum_{k=0}^{N}B_k^N(\theta)b_k^N\Big)\,d\theta\right| \\ &\le |b_0^N|\,\|A\|_{L^2}\|\varphi^N\|_{L^2} \le \|A\|^2_{L^2}|b_0^N|^2 + \frac{1}{4}\|\varphi^N\|^2_{L^2} \\ &\le \|A\|^2_{L^2}|b_0^N|^2 + \frac{r}{8N}\sum_{k-1}^{N}\left(|b_{k-1}^N|^2 + |b_k^N|^2\right). \end{aligned}$$

The estimates (3.8) and (3.10) together with (3.7) imply

$$\begin{aligned} \langle \mathcal{A}^N z, z\rangle_N &\le \omega|b_0^N|^2 - \frac{r}{2N}\sum_{k=1}^{N}|b_k^N|^2 + \frac{r}{8N}\sum_{k=1}^{N}\left(|b_{k-1}^N|^2 + |b_k^N|^2\right) \\ &\le \omega|b_0^N|^2 - \frac{r}{4N}\sum_{k=1}^{N}\left(|b_{k-1}^N|^2 + |b_k^N|^2\right) \\ &\quad + \frac{r}{4N}|b_0^N|^2 - \frac{r}{4N}|b_N^N|^2 + \frac{r}{8N}\sum_{k=1}^{N}\left(|b_{k-1}^N|^2 + |b_k^N|^2\right) \\ &\le \omega_N|b_0^N|^2 - \frac{r}{8N}\sum_{k=1}^{N}\left(|b_{k-1}^N|^2 + |b_k^N|^2\right). \end{aligned}$$

Observing (3.9) and $b_0^N = \eta$ we get the desired result. ∎

Remarks. 1. It is not difficult to verify that

$$(3.11) \qquad \langle \mathcal{A}z, z\rangle_g \le \omega|\varphi(0)|^2 - \frac{1}{4}\|\varphi\|^2_{L^2} \quad \text{for all } z = (\varphi(0),\varphi) \in \operatorname{dom}\mathcal{A}.$$

2. In the estimate (3.10) the right-hand side can be replaced by $(1/2)\|A\|^2_{L^2}|b_0^N|^2 + (r/4N)\sum_{k=1}^{N}\left(|b_{k-1}^N|^2 + |b_k^N|^2\right)$ which would give the dissipativity estimate

$$\langle \mathcal{A}^N z, z\rangle_N \le \left(\frac{r}{N} + \frac{1}{2}\Big(\lambda_{\max}(A_0 + A_0^{\mathrm{T}}) + \ell + r + \sum_{j=1}^{\ell}|A_j|^2 + \|A\|^2_{L^2}\Big)\right)|\eta|^2.$$

This would also be sufficient for b). On the other hand, it was shown in [11] that the estimate as given in a) can be used to give a proof for uniform exponential stability of the approximation scheme different from that in the present paper (which is based on uniform differentiability of the approximating semigroups).

3.2. Consistency of the scheme in Z. As we shall see it is considerably simpler to verify assumption (ii) of Theorem 3.1 for $\mathcal{A}^N$ and $\mathcal{A}$ if $0 \in \rho(\mathcal{A})$. In this case obviously also $0 \in \rho(\mathcal{A}^N)$ for all N because $\Delta^N(0) = \Delta(0)$. In this case we also get assumption (ii) for $(\mathcal{A}^N)^*$ and $\mathcal{A}^*$ without additional efforts.

LEMMA 3.3. *Suppose $0 \in \rho(\mathcal{A})$. Then*

$$\lim_{N\to\infty} \|(\mathcal{A}^N)^{-1}P^N z - P^N\mathcal{A}^{-1}z\|_N = 0$$

and

$$\lim_{N\to\infty} \|((\mathcal{A}^N)^*)^{-1}P^N z - P^N(\mathcal{A}^*)^{-1}z\|_N = 0$$

for all $z \in Z$.

PROOF: The adjoint of P^N is the injection $i^N : Z^N \to Z$. Then observing $\Delta^N(0) = \Delta(0)$ and $e_0^N(\theta) \equiv 1$ we get from Proposition 2.3,b) that $(\mathcal{A}^N)^{-1}z = P^N\mathcal{A}^{-1}z$ for $z \in Z^N$, i.e., $(\mathcal{A}^N)^{-1} = P^N\mathcal{A}^{-1}i^N$ and $((\mathcal{A}^N)^*)^{-1} = P^N(\mathcal{A}^*)^{-1}i^N$. Therefore for any $z \in Z$

$$\|(\mathcal{A}^N)^{-1}P^N z - P^N\mathcal{A}^{-1}z\|_N = \|P^N\mathcal{A}^{-1}(P^N z - z)\|_N.$$

The result follows from (K 1) and $P^N z \to z$. The proof for the adjoint operators is completely analogous. ∎

The basic convergence result for our scheme in the state space Z is contained in

THEOREM 3.4. *Let $S^N(t)$, $t \geq 0$, be the semigroup on Z^N generated by $\mathcal{A}^N$, i.e., $S^N(t) = e^{\mathcal{A}^N t}$, $t \geq 0$. Then for all $z \in Z$*

$$\lim_{N\to\infty} S^N(t)P^N z = S(t)z \quad \text{and} \quad \lim_{N\to\infty} S^N(t)^*P^N z = S(t)^*z$$

uniformly for t in bounded intervals.

PROOF: Let us define the operators $\mathcal{T}: Z \to Z$ and $\mathcal{T}^N: Z^N \to Z^N$ by

(3.12) $$\mathcal{T}(\eta,\varphi) = (\eta, 0) \quad \text{for } (\eta,\varphi) \in Z \text{ and } \quad \mathcal{T}^N = P^N\mathcal{T}|_{Z^N}.$$

Replacing the operators $\mathcal{A}$ and $\mathcal{A}^N$ by $\mathcal{A} - \kappa\mathcal{T}$ and $\mathcal{A}^N - \kappa\mathcal{T}^N$, respectively, just means that the matrix A_0 in (1.1) is replaced by $A_0 - \kappa I_n$. Therefore we can choose $\kappa \in \mathbf{R}$ such that the constants ω and ω^N corresponding to $\mathcal{A} - \kappa\mathcal{T}$ and $\mathcal{A}^N - \kappa\mathcal{T}^N$ (where ω and ω^N are defined in Lemma 3.2), respectively, are negative. Let $T(t)$ and $T^N(t)$ be the semigroups generated by $\mathcal{A} - \kappa\mathcal{T}$ and $\mathcal{A}^N - \kappa\mathcal{T}^N$. By Lemmas 3.2 and 3.3 the assumptions of Theorem 3.1 are satisfied for $T(t)$ and $T^N(t)$ with $\lambda = 0$ in assumption (ii). Hence for all $z \in Z$

$$\lim_{N\to\infty} \|T^N(t)P^N z - P^N T(t)z\|_N = 0$$

uniformly on bounded t-intervals. Since the norms $\|\cdot\|_N$ and $\|\cdot\|$ are uniformly equivalent and $\|P^N T(t)z - T(t)z\| \to 0$ as $N \to \infty$ uniformly on bounded t-intervals, we obtain

$$\lim_{N\to\infty} T^N(t)P^N z = T(t)z \tag{3.13}$$

uniformly on bounded intervals.

Returning to the semigroups $S(t)$ and $S^N(t)$ we first observe that by the variation of constants formula

$$\begin{aligned} S(t) &= T(t) + \kappa \int_0^t T(t-s)\mathcal{T} S(s)\, ds, \\ S^N(t) &= T^N(t) + \kappa \int_0^t T^N(t-s)\mathcal{T}^N S^N(s)\, ds, \end{aligned} \tag{3.14}$$

for $t \geq 0$. Since $\mathcal{T}$ is a finite rank operator, (3.13) implies

$$\|T^N(t)P^N\mathcal{T} - T(t)\mathcal{T}\|_{\mathcal{L}(Z,Z)} \to 0 \quad \text{as } N \to \infty \tag{3.15}$$

uniformly on bounded t-intervals. From (3.14) we get

$$\begin{aligned} \|S(t)z - S^N(t)P^N z\| \leq{} & \|T(t)z - T^N(t)P^N z\| \\ & + \kappa \int_0^t \|T(t-s)\mathcal{T} - T^N(t-s)P^N\mathcal{T}\|_{\mathcal{L}(Z,Z)} \|S^N(s)P^N z\|\, ds \\ & + \kappa \int_0^t \|T(t-s)\mathcal{T}\|_{\mathcal{L}(Z,Z)} \|S(s)z - S^N(s)P^N z\|\, ds. \end{aligned}$$

By Gronwall's lemma using Lemma 3.2. (3.13) and (3.15) we obtain

$$\lim_{N\to\infty} S^N(t)P^N z = S(t)z$$

uniformly on bounded t-intervals for all $z \in Z$.

The proof for the adjoint semigroups is completely analogous. ∎

COROLLARY 3.5. *For $z \in Z$ and $u \in L^2_{\text{loc}}(0,\infty;\mathbf{R}^m)$ let $z^N(t)$ be the solution of (2.3) with $z^N(0) = P^N z$ and $z(t)$ be the solution of (1.3) with $z(0) = z$. Then for any $T > 0$*

$$\lim_{N\to\infty} z^N(t) = z(t)$$

uniformly for $t \in [0,T]$ and uniformly for $u|_{[0,T]}$ in bounded subsets of $L^1(0,T;\mathbf{R}^m)$.

PROOF: On the basis of Theorem 3.4 the proof is quite analogous to the proof given in [2] for a similar result. ∎

Remark. In Appendix B we shall prove that $(\lambda I - \mathcal{A}^N)^{-1}P^N z \to P^N(\lambda I - \mathcal{A})^{-1}z$ as $N \to \infty$ for all $z \in Z$ and any $\lambda \in \rho(\mathcal{A})$ (Lemma B.1). In view of this result the introduction of the operators $\mathcal{T}$ and $\mathcal{T}^N$ in the proof of Theorem 3.4 is in principle not necessary as far as convergence of the semigroups $S^N(t)$ in Z is of concern. But we already have seen in this section that the approach taken in this section gives also convergence for the adjoint semigroups $S^N(t)^*$ without additional efforts. On the other hand the proof of $\left(\lambda I - (\mathcal{A}^N)^*\right)^{-1} P^N z \to P^N(\lambda I - \mathcal{A}^*)^{-1}z$ for $\lambda \neq 0$ would be very involved.

3.3. Convergence of the scheme in H^1. We shall prove convergence of the scheme in H^1 using results already obtained for the scheme in Z. For the nonhomogeneous problem we shall also need

LEMMA 3.6. *Let $w^N(t)$, $t \geq 0$, be the solution of (2.2). Then there exists a constant $\tilde{\omega} > 0$ (in fact $\tilde{\omega} = (3/2)\|L\|_{\mathcal{L}(H^1,\mathbf{R}^n)} + 1$) such that for all N*

$$\|\iota w^N(t)\|^2_{H^1} \leq e^{2\tilde{\omega}t}\|\iota w^N(0)\|^2_{H^1} + 3\int_0^t e^{2\tilde{\omega}(t-s)}|B|^2|u(s)|^2 ds, \quad t \geq 0.$$

PROOF: Let $\iota w^N(t) = B^N b^N(t)$. Then

$$\|\iota w^N(t)\|^2_{H^1} = |b_0^N(t)|^2 + \frac{N}{r}\sum_{k=1}^N |b_{k-1}^N(t) - b_k^N(t)|^2.$$

Since $b^N(t)$ solves equation (2.5), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\iota w^N(t)\|^2_{H^1} &= \dot{b}_0^N(t)^{\mathrm{T}} b_0^N(t) + \frac{N}{r}\sum_{k=1}^N \left(\dot{b}_{k-1}^N(t) - \dot{b}_k^N(t)\right)^{\mathrm{T}} \left(b_{k-1}^N(t) - b_k^N(t)\right) \\
&= \Big(\sum_{k=0}^N D_k^N b_k^N(t) + Bu(t)\Big)^{\mathrm{T}} b_0^N(t) + \frac{1}{2}\sum_{k=1}^N \left(\dot{b}_{k-1}^N(t) - \dot{b}_k^N(t)\right)^{\mathrm{T}} \left(\dot{b}_{k-1}^N(t) + \dot{b}_k^N(t)\right) \\
&= \left(L(\iota w^N(t)) + Bu(t)\right)^{\mathrm{T}} b_0^N(t) + \frac{1}{2}|\dot{b}_0^N(t)|^2 - \frac{1}{2}|\dot{b}_N^N(t)|^2 \\
&\leq \frac{1}{2}|L(\iota w^N(t))|^2 + \frac{1}{2}|B|^2|u(t)|^2 + |b_0^N(t)|^2 + |L(\iota w^N(t))|^2 + |B|^2|u(t)|^2 \\
&\leq \left(\frac{3}{2}\|L\|_{\mathcal{L}(H^1,\mathbf{R}^n)} + 1\right)\|\iota w^N(t)\|^2_{H^1} + \frac{3}{2}|B|^2|u(t)|^2.
\end{aligned}$$

Thus Gronwall's inequality yields the result. ∎

THEOREM 3.7. *For $z = (\varphi(0), \varphi)$ with $\varphi \in H^1$ and $u \in L^2_{\mathrm{loc}}(0,\infty;\mathbf{R}^m)$ let $z(t)$ be given by (1.4) and $w^N(t)$ be the solution of (2.2) with initial value $w^N(0) = \iota P_1^N \varphi$. Then*

$$\lim_{N\to\infty} \|\iota w^N(t) - \iota z(t)\|_{H^1} = 0$$

uniformly on bounded t-intervals.

PROOF: Since H^1 and $\operatorname{dom}\mathcal{A}$ supplied with the graph norm are metrically isomorphic, it suffices to prove $\|\mathcal{A}w^N(t) - \mathcal{A}z(t)\|_Z \to 0$ and $(\iota w^N(t))(0) \to (\iota z(t))(0)$ as $N \to \infty$ uniformly on bounded t-intervals. By definition of $\mathcal{A}^N$ and L^N we have

$$\mathcal{A}^N P^N w^N(t) = \mathcal{A}L^N P^N w^N(t) = \mathcal{A}w^N(t), \quad t \geq 0. \tag{3.16}$$

We first consider the case $u \equiv 0$. Using (3.16) we obtain from (2.2) $(d/dt)\mathcal{A}w^N(t) = \mathcal{A}^N(d/dt)P^N w^N(t) = \mathcal{A}^N \mathcal{A}w^N(t)$, i.e.,

$$\mathcal{A}w^N(t) = S^N(t)\mathcal{A}w^N(0), \quad t \geq 0. \tag{3.17}$$

Lemma 2.1 implies

$$P^N \mathcal{A}z(0) - \mathcal{A}w^N(0) = \big(L(\varphi - \varphi^N), 0\big), \tag{3.18}$$

where $\varphi^N = \sum_{j=0}^N B_k^N \varphi(t_k^N)$. For some $K > 0$ independent of N we have (cf. [23, Exercise (2.10)])

$$\|\varphi^N - \varphi\|_{L^\infty} \leq K(r/N)^{1/2}\|\varphi\|_{H^1} \tag{3.19}$$

for $N = 1, 2, \ldots$. This together with (3.17) and (3.18) implies

$$\begin{aligned}
&\|\mathcal{A}w^N(t) - \mathcal{A}z(t)\|_Z \\
&\qquad \leq \|S^N(t)\|\, \|\mathcal{A}w^N(0) - P^N\mathcal{A}z(0)\|_Z + \|S^N(t)P^N\mathcal{A}z(0) - S(t)\mathcal{A}z(0)\|_Z \\
&\qquad \leq Me^{\omega t}\|L\|_{\mathcal{L}(H^1, \mathbf{R}^n)} K\Big(\frac{r}{N}\Big)^{1/2}\|\varphi\|_{H^1} \\
&\qquad\qquad + \|S^N(t)P^N\mathcal{A}z(0) - S(t)\mathcal{A}z(0)\|_Z \quad \text{for } t \geq 0.
\end{aligned}$$

By Theorem 3.4 this gives

$$\lim_{N \to \infty} \|\mathcal{A}w^N(t) - \mathcal{A}z(t)\|_Z = 0$$

uniformly for t in bounded intervals. Since $z^N(t) = P^N w^N(t)$, $t \geq 0$, satisfies (2.3), by Theorem 3.4 we also get $\lim_{N\to\infty} \|P^N w^N(t) - z(t)\|_Z = 0$ uniformly for t in bounded intervals. This implies

$$|(\iota w^N(t))(0) - (\iota z(t))(0)| \to 0 \quad \text{as} \quad N \to \infty$$

uniformly on bounded t-intervals. Note that $(\iota w^N(t))(0)$ and $(\iota z(t))(0)$ are the vector components of $P^N w^N(t)$ and $z(t)$, respectively.

We next consider the case $z = 0$ and $u \in C^1(0, T; \mathbf{R}^m)$, $T > 0$. Equation (2.2) together with (3.16) implies

$$P^N w^N(t) = \int_0^t S^N(t-s)\mathcal{B}u(s)\,ds, \quad t \geq 0.$$

Hence we have

$$\mathcal{A}w^N(t) = \mathcal{A}^N P^N w^N(t) = \mathcal{A}^N \int_0^t S^N(t-s)\mathcal{B}u(s)\,ds$$
$$= S^N(t)\mathcal{B}u(0) - \mathcal{B}u(t) + \int_0^t S^N(t-s)\mathcal{B}\dot{u}(s)\,ds.$$

Similarly we get

$$\mathcal{A}z(t) = S(t)\mathcal{B}u(0) - \mathcal{B}u(t) + \int_0^t S(t-s)\mathcal{B}\dot{u}(s)\,ds.$$

Therefore

$$\mathcal{A}z(t) - \mathcal{A}w^N(t) = \big(S(t) - S^N(t)\big)\mathcal{B}u(0) + \int_0^t \big(S(t-s) - S^N(t-s)\big)\mathcal{B}\dot{u}(s)\,ds.$$

By Corollary 3.5 this implies

$$\lim_{N\to\infty} \|\mathcal{A}z(t) - \mathcal{A}w^N(t)\|_Z = 0$$

uniformly for t in bounded intervals. As in the previous case (but now using Corollary 3.5) we also get $\lim_{N\to\infty}(\iota w^N(t))(0) = (\iota z(t))(0)$ uniformly on bounded t-intervals.

In case $z = 0$ and $u \in L^2(0,T;\mathbf{R}^m)$ we have the estimate (see Lemma 3.6)

$$\|\iota w^N(t)\|_{H^1} \le \sqrt{3}e^{\tilde{\omega}t}|B|\,\|u\|_{L^2(0,T;\mathbf{R}^m)}, \quad 0 \le t \le T.$$

We have an analogous estimate for $\iota z(t)$. Density of $C^1(0,T;\mathbf{R}^m)$ in $L^2(0,T;\mathbf{R}^m)$ and a simple application of the triangle inequality implies the result for the general case. ∎

4. Uniform differentiablity

In this section we first establish differentiability of the semigroups $S^N(t)$ uniformly with respect to N. Uniform differentiability of the approximating semigroups $S^N(t)$ is fundamental for our approach to obtain rates of convergence for our scheme (see Section 6). Also the proof for the uniform exponential stability property of the approximating semigroups (see Section 5) is based on uniform differentiability of the approximating semigroups though a different proof would also be possible (see the remarks at the end of Section 3.1). For a fixed real number $b > r$ and numbers $a > 0$, $\omega > 0$ we define

$$\Sigma_{a,\omega} = \{\lambda \in \mathbf{C} \mid |\mathrm{Im}\,\lambda| \geq e^{(a-\mathrm{Re}\,\lambda)b} \quad \text{and} \quad \mathrm{Re}\,\lambda \leq \omega\}$$

and the "exponential sector"

$$\mathcal{S}_{a,\omega} = \{\lambda \in \mathbf{C} \mid \mathrm{Re}\,\lambda \leq \omega \text{ and } \lambda \notin \Sigma_{a,\omega}\}.$$

THEOREM 4.1. *a) Let $\omega > \gamma$ (γ given in Lemma 2.4), $b > r$, $0 < \delta < 1$ and put $a_1 = \omega + \frac{1}{b}\ln\frac{\omega}{1-\delta}$. Then for all $a \geq \max(a_0, a_1)$ (a_0 as defined in Lemma A.5)*

$$\sigma(\mathcal{A}^N) \subset \mathcal{S}_{a,\omega} \quad \text{for all } N$$

and there exists a positive constant c_1 such that

$$\|(\lambda I - \mathcal{A}^N)^{-1}\| \leq c_1|\mathrm{Im}\,\lambda|$$

for all $\lambda \in \Sigma_{a,\omega}$ and all N.

b) For any compact subset K of $\rho(\mathcal{A})$ there exist positive constants c_2 and N_2 such that $K \subset \rho(\mathcal{A}^N)$ for $N \geq N_2$ and

$$\|(\lambda I - \mathcal{A}^N)^{-1}\| \leq c_2 \quad \text{for } \lambda \in K \text{ and } N \geq N_2.$$

PROOF: By definition of a_1 and K_0 we have

$$K_0 e^{-ab} \leq \frac{1-\delta}{\omega} \quad \text{and} \quad |\mathrm{Im}\,\lambda| \geq \frac{\omega}{1-\delta} \text{ for } \lambda \in \Sigma_{a,\omega}. \tag{4.1}$$

We first prove that $\lambda \in \rho(\mathcal{A}^N)$ for all $\lambda \in \Sigma_{a,\omega}$ and all N. According to Theorem 2.3.a) this is equivalent to $\det \Delta^N(\lambda) \neq 0$ for $\lambda \in \Sigma_{a,\omega}$. Let $a \geq \max(a_0, a_1)$. Then Lemma A.5 and (4.1) imply

$$\begin{aligned}|L(\epsilon_\lambda^N I_n)| &\leq \gamma \max_{0\leq k\leq N} |\mu(\lambda r/N)|^k \leq \omega K_0 e^{-ab}|\mathrm{Im}\,\lambda| \\ &\leq (1-\delta)|\lambda| \quad \text{for } \lambda \in \Sigma_{a,\omega},\ N = 1, 2, \dots\ .\end{aligned}$$

Therefore $|\Delta^N(\lambda)| \geq \delta|\lambda|$ for $\lambda \in \Sigma_{a,\omega}$, $N = 1, 2, \dots$, and (taking the Neuman series for $\Delta^N(\lambda)^{-1}$)

$$|\Delta^N(\lambda)^{-1}| \leq \frac{1}{\delta|\lambda|} \quad \text{for } \lambda \in \Sigma_{a,\omega},\ N = 1, 2, \dots\ . \tag{4.2}$$

This proves a). Note that by Lemma 2.4 any λ with $\operatorname{Re}\lambda > \gamma$ is in $\rho(\mathcal{A}^N)$ for all N. According to Theorem 2.3,b)

$$(\lambda I - \mathcal{A}^N)^{-1} z = P^N(\psi(0), \psi) \quad \text{for all } z \in Z^N,$$

where ψ is given in Theorem 2.3 and $z = (\eta, \varphi) = \hat{E}^N a^N$. Using Lemma A.5 we get ($\mu = \mu(r\lambda/N)$)

$$
\begin{aligned}
|r^N(t_k^N)| &= \Big|\frac{1}{2}\frac{r}{N}(1+\mu)\sum_{j=1}^{k}\mu^{k-j}a_j^N\Big| \\
&\le \frac{1}{2}\frac{r}{N}\Big(|a_1^N| + \sum_{j=1}^{k-1}(|a_j^N| + |a_{j+1}^N|) + |a_k^N|\Big)\max_{0\le j\le k}|\mu|^j \\
&\le \frac{r}{N}\Big(\sum_{j=1}^{k}|a_j^N|\Big)K_0 e^{-ab}|\operatorname{Im}\lambda| \qquad (4.3) \\
&\le \Big(\frac{kr}{N}\Big)^{1/2}\Big(\frac{r}{N}\sum_{j=1}^{k}|a_j^N|^2\Big)^{1/2}\frac{1-\delta}{\omega}|\operatorname{Im}\lambda| \\
&\le r^{1/2}\frac{1-\delta}{\omega}\|\varphi\|_{L^2}|\operatorname{Im}\lambda|
\end{aligned}
$$

and

$$|L(r^N)| \le \frac{\gamma}{\omega} r^{1/2}(1-\delta)\|\varphi\|_{L^2}|\operatorname{Im}\lambda| \tag{4.4}$$

for all $\lambda \in \Sigma_{a,\omega}$ and all N. The estimates (4.1), (4.2) and (4.4) imply

$$
\begin{aligned}
|\psi(0)| &\le \frac{1}{\delta|\lambda|}\Big(|a_0^N| + \gamma r^{1/2}\frac{1-\delta}{\omega}\|\varphi\|_{L^2}|\operatorname{Im}\lambda|\Big) \\
&\le \frac{1}{\delta|\lambda|}\Big(\frac{1}{\omega} + r^{1/2}\|\varphi\|_{L^2}\Big)(1-\delta)|\operatorname{Im}\lambda| \qquad (4.5) \\
&\le \frac{1-\delta}{\delta}\Big(\frac{1}{\omega} + r^{1/2}\Big)\|z\|
\end{aligned}
$$

for all $\lambda \in \Sigma_{a,\omega}$, $z \in Z^N$ and $N = 1, 2, \ldots$. Using (4.1) again we also have

$$|\psi(0)| \le \frac{(1-\delta)^2}{\omega\delta}\Big(\frac{1}{\omega} + r^{1/2}\Big)\|z\|\,|\operatorname{Im}\lambda| \tag{4.6}$$

for all $\lambda \in \Sigma_{a,\omega}$, $z \in Z^N$ and $N = 1, 2, \ldots$. From (4.3)-(4.5) and Lemma A.5 we obtain

$$|\psi(t_k^N)| \le \frac{1-\delta}{\omega}\Big(\frac{1-\delta}{\delta}\big(\frac{1}{\omega} + r^{1/2}\big) + r^{1/2}\Big)\|z\|\,|\operatorname{Im}\lambda|, \quad k = 1, \ldots, N. \tag{4.7}$$

for all $\lambda \in \Sigma_{a,\omega}$, $z \in Z^N$ and $N = 1, 2, \dots$. Let $\tilde{c}_1 = \frac{1-\delta}{\omega}\Big(\frac{1-\delta}{\delta}(\frac{1}{\omega} + r^{1/2}) + r^{1/2}\Big)$. Then (compare also (3.9)) by (4.6) and (4.7)

$$\|P^N(\psi(0), \psi)\|^2 \le |\psi(0)|^2 + \frac{r}{2N}\sum_{k=1}^{N}\big(|\psi(t_{k-1}^N)|^2 + |\psi(t_k^N)|^2\big)$$
$$\le \tilde{c}_1^2(1+r)\|z\|^2|\operatorname{Im}\lambda|^2$$

for all $\lambda \in \Sigma_{a,\omega}$, $z \in Z^N$ and $N = 1, 2, \dots$.

In order to prove b) we use Corollary A.4 to see that $K \subset \rho(\mathcal{A}^N)$ for N sufficiently large, say $N \ge N_2$ (compare Proposition 2.3,a)). We also see that $\Delta^N(\lambda)^{-1} \to \Delta(\lambda)^{-1}$ uniformly on K, which implies $|\Delta^N(\lambda)^{-1}| \le$const. for $\lambda \in K$ and $N \ge N_2$. Furthermore Corollary A.3 shows that $\sup_{-r\le\theta\le 0}|e_\lambda^N(\theta)| = \max_{0\le j\le N}|\mu(r\lambda/N)|^j \le$ const. for $\lambda \in K$ and all N. Therefore (4.3) reduces to $|r^N(t_k^N)| \le$ const.$\|\varphi\|_{L^2}$ for $\lambda \in K$ and all N. The rest of the proof uses the correspondingly simplified versions of (4.4)-(4.7) in order to get

$$\|P^N(\psi(0), \psi)\|^2 \le \text{const.}\|z\|^2 \quad \text{for } \lambda \in K \text{ and } N \ge N_1. \quad \blacksquare$$

It is easy to see that for arbitrary $a, \omega \in \mathbf{R}$ and $b > r$ we have

$$|e^{\lambda\theta}| \le K_0 e^{-ab}|\operatorname{Im}\lambda|, \quad -r \le \theta \le 0, \quad \lambda \in \Sigma_{a,\omega}. \tag{4.8}$$

If we choose $\omega > \gamma$, $a \ge \max(a_0, a_1)$, then analogous computations as in the proof of Theorem 4.1 show that

$$\|(\lambda I - \mathcal{A})^{-1}\| \le c_1|\operatorname{Im}\lambda| \quad \text{for all } \lambda \in \Sigma_{a,\omega}.$$

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda \le \omega \text{ and } \operatorname{Im}\lambda = e^{(a-\operatorname{Re}\lambda)b}\},$$
$$\Gamma_2 = \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda = \omega \text{ and } |\operatorname{Im}\lambda| \le e^{(a-\omega)b}\},$$
$$\Gamma_3 = \{\lambda \in \mathbf{C} \mid \bar{\lambda} \in \Gamma_1\}.$$

The following theorem is an immediate consequence from [20; Thm. 4.7].

THEOREM 4.2. *a) For all N and all $b > r$*

$$S^N(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - \mathcal{A}^N)^{-1}\,d\lambda, \quad t > 2b, \tag{4.9}$$

and

$$\mathcal{A}^N S^N(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}(\lambda I - \mathcal{A}^N)^{-1}\,d\lambda, \quad t > 3b. \tag{4.10}$$

b) For all $b > r$

$$S(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - \mathcal{A})^{-1}d\lambda, \quad t > 2b, \tag{4.11}$$

and

$$\mathcal{A}S(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}(\lambda I - \mathcal{A})^{-1}d\lambda, \quad t > 3b. \tag{4.12}$$

Remark. The contents of Theorems 4.1 and 4.2 explain why we labeled our scheme "uniformly differentiable".

5. Uniform exponential stability

In order to establish the property of uniform exponential stability for our scheme we shall use convergence of the resolvent operators $(\lambda I - \mathcal{A}^N)^{-1}$. Our approach is the same as in [19].

THEOREM 5.1. *Let* $\omega_0 = \sup\{Re\,\lambda \mid \lambda \in \sigma(\mathcal{A})\}$. *Then for any* $\epsilon > 0$ *there exists a constant* $M = M(\epsilon)$ *such that*

$$\|S^N(t)\| \le M(\epsilon)e^{(\omega_0+\epsilon)t}, \quad t \ge 0,$$

for N *sufficiently large.*

PROOF: By assumption we have $\lambda \in \rho(\mathcal{A})$ for $\mathrm{Re}\,\lambda > \omega_0$. Corollary A.4 together with Proposition 2.3,a) implies that for N sufficiently large $\lambda \in \rho(\mathcal{A}^N)$ for $\mathrm{Re}\,\lambda \ge \omega_0 + \epsilon$. Therefore we can choose $\omega = \omega_0 + \epsilon$ in the definition of the path Γ for the representation (4.9). This is a consequence of Cauchy's theorem, because $(\lambda I - \mathcal{A})^{-1}$ and $(\lambda I - \mathcal{A}^N)^{-1}$ for N sufficiently large are analytic in $\mathrm{Re}\,\lambda \ge \omega_0 + \epsilon$. Furthermore we choose $b = 5r/4$. From (4.9) we get the estimate

$$\begin{aligned}\|S^N(t)\| &\le \frac{1}{2\pi}\int_{\Gamma_1\cup\Gamma_3} e^{t\mathrm{Re}\,\lambda}\|(\lambda I - \mathcal{A}^N)^{-1}\|\,|d\lambda| \\ &\quad + \frac{1}{2\pi}e^{(\omega_0+\epsilon)t}\int_{\Gamma_2}\|(\lambda I - \mathcal{A}^N)^{-1}\|\,|d\lambda| =: J_1 + J_2 \quad \text{for } t > 2b.\end{aligned}$$

Using Theorem 4.1,a) the estimate for J_1 is

$$J_1 \le \frac{c_1}{\pi}\int_{\Gamma_1} e^{t\mathrm{Re}\,\lambda}|\mathrm{Im}\,\lambda|\,|d\lambda|, \quad t > 2b.$$

A parameter representation for Γ_1 is

$$\lambda(\tau) = \tau + ie^{(a-\tau)b}, \quad -\infty < \tau \le \omega_0 + \epsilon, \tag{5.1}$$

with $|\dot\lambda(\tau)| = \left(1 + b^2e^{2(a-\tau)b}\right)^{1/2} \le e^{-\tau b}\left(e^{(\omega_0+\epsilon)b} + b^2e^{2ab}\right)^{1/2}$. Therefore

$$J_1 \le \text{const.}\int_{-\infty}^{\omega_0+\epsilon} e^{\tau(t-2b)}d\tau \le \frac{\text{const.}}{t-2b}e^{(\omega_0+\epsilon)t} \le ce^{(\omega_0+\epsilon)t} \quad \text{for } t \ge 3r.$$

where c is appropriately chosen (note that $2b = 5r/2 < 3r$).

The path Γ_2 is a compact subset of $\rho(\mathcal{A})$. By Theorem 4.1,b) we have for $\lambda \in \Gamma_2$ and N sufficiently large

$$J_2 \le \text{const.}e^{(\omega_0+\epsilon)t} \quad \text{for } t \ge 3r.$$

which together with the estimate for J_1 shows that

$$\|S^N(t)\| \le \text{const.}e^{(\omega_0+\epsilon)t} \quad \text{for } t \ge 3r. \tag{5.2}$$

By Lemma 3.2 $\left(\|S^N(t)\|\right)_{N=1,2,\ldots}$ is uniformly bounded on $[0, 3r]$. This together with (5.2) proves the result. ∎

Remark. If $\omega_0 < 0$ then Theorem 5.3 states that the approximating semigroups $S^N(t)$ are exponentially stable for N sufficiently large with a uniform decay rate, which can be chosen arbitrarily close (from above) to the decay rate of the semigroup $S(t)$.

6. Convergence rates

In this section we establish rates of convergence for our scheme. The basic idea for the approach, namely to use uniform differentiability of the approximating semigroups, was introduced by I. Lasiecka and A. Manitius in [19] for the scheme of averaging approximations. We shall prove two kinds of rate estimates: a) Optimal rates on $t \geq 0$ for sufficiently smooth initial data. b) Optimal rates on $t \geq t_0 > 0$ for general non-smooth data.

For fixed $\omega > \gamma$ (γ as defined in Lemma 2.4) and $\epsilon > 0$ we choose $b \in (r, r+\epsilon/7)$ and $a = \max(a_0, a_1)$ (a_0 defined in Lemma A.4 and a_1 in Theorem 4.1,a)) in the definition of $\Sigma_{a,\omega}$. Let $\omega_0 = \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathcal{A})\}$. For any $\omega_1 \in (\omega_0, \omega]$ the path $\Gamma_{\omega_1} = \Gamma_{1,\omega_1} \cup \Gamma_{2,\omega_1} \cup \Gamma_{3,\omega_1}$ is defined by

$$\begin{aligned}
\Gamma_{1,\omega_1} &= \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda \leq \omega_1 \text{ and } \operatorname{Im}\lambda = e^{(a-\operatorname{Re}\lambda)b}\},\\
\Gamma_{2,\omega_1} &= \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda = \omega_1 \text{ and } |\operatorname{Im}\lambda| \leq e^{(a-\omega_1)b}\},\\
\Gamma_{3,\omega_1} &= \{\lambda \in \mathbf{C} \mid \bar{\lambda} \in \Gamma_{1,\omega_1}\}.
\end{aligned}$$

The following assumption will be used in some of the estimates:

(H) The delays θ_j, $j = 1, \dots, \ell$, are commensurate (i.e., for $\hat{r} > 0$ and natural numbers k_j we have $\theta_j = k_j\hat{r}$, $j = 1, \dots, \ell$) and are contained in the set of meshpoints t_k^N, $k = 1, \dots, N$ (which is the case for $N = mk_\ell$, $m = 1, 2, \dots$).

The first result on convergence rates is concerned with norm convergence of the approximating semigroups (with rates $O(1/N)$ resp. $O(1/N^{1/2})$) for t sufficiently large and with convergence uniformly on bounded intervals if initial data are in $\operatorname{dom}\mathcal{A}^2$.

THEOREM 6.1. *Let $\alpha = 1$ if (H) is satisfied, otherwise $\alpha = 1/2$.*
a) For any $\epsilon > 0$ there exist positive constants c_1, c_2 and N_1 such that for $N \geq N_1$

$$\|S^N(t)P^N - P^N S(t)\|_{\mathcal{L}(Z,Z)} \leq c_1\Big(\frac{r}{N}\Big)^\alpha e^{(\omega_0+\epsilon)t}, \quad t \geq 5r + \epsilon. \tag{6.1}$$

and

$$\|L^N S^N(t)P^N - S(t)\|_{\mathcal{L}(Z,\operatorname{dom}\mathcal{A})} \leq c_2\Big(\frac{r}{N}\Big)^\alpha e^{(\omega_0+\epsilon)t}, \quad t \geq 6r + \epsilon. \tag{6.2}$$

b) For any $T > 0$ there exist positive constants c_3 and N_2 such that for $N \geq N_2$

$$\|S^N(t)P^N z - P^N S(t)z\| \leq c_3\Big(\frac{r}{N}\Big)^\alpha \|z\|_{\operatorname{dom}\mathcal{A}^2}, \quad 0 \leq t \leq T, \tag{6.3}$$

for all $z \in \operatorname{dom}\mathcal{A}^2$ ($\|z\|_{\operatorname{dom}\mathcal{A}^2} = \|z\| + \|\mathcal{A}^2 z\|$ for $z \in \operatorname{dom}\mathcal{A}^2$).

PROOF: For $\epsilon > 0$ and sufficiently small let $\omega_1 = \omega_0 + \epsilon$. By Theorem 4.1,a) we have $\sigma(\mathcal{A}^N) \subset S_{a,\omega}$ for all N. From Theorem 5.1 we infer that $\operatorname{Re}\lambda < \omega_1$ for all $\lambda \in \sigma(\mathcal{A}^N)$

and N sufficiently large, say $N \geq N_1$. By Cauchy's theorem we can replace Γ in Theorem 4.2 by Γ_{ω_1} for $N \geq N_1$. Using (4.9) and (4.11) we obtain

$$\begin{aligned}\|P^N S(t) - S^N(t)P^N\| &\leq \frac{1}{2\pi}\Big\|\int_{\Gamma_{1,\omega_1}\cup\Gamma_{3,\omega_1}} e^{\lambda t}\Big(P^N(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{A}^N)^{-1}P^N\Big)\,d\lambda\Big\| \\ &\quad + \frac{1}{2\pi}\Big\|\int_{\Gamma_{2,\omega_1}} e^{\lambda t}\Big(P^N(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{A}^N)^{-1}P^N\Big)\,d\lambda\Big\| \\ &=: \frac{1}{2\pi}(J_1 + J_2).\end{aligned}$$

Using the parameter representation (5.1) for Γ_{1,ω_1} and Lemma B.1,a) we get (note that $5b < 5r + \epsilon$)

$$\begin{aligned}J_1 &\leq 2c\big(\frac{r}{N}\big)^\alpha \int_{\Gamma_{1,\omega_1}} e^{t\mathrm{Re}\,\lambda}|\mathrm{Im}\,\lambda|^4|d\lambda| \\ &\leq \mathrm{const.}\big(\frac{r}{N}\big)^\alpha \int_{-\infty}^{\omega_0+\epsilon} e^{\tau(t-5b)}d\tau \leq \mathrm{const.}\big(\frac{r}{N}\big)^\alpha e^{(\omega_0+\epsilon)t}\end{aligned}$$

for $t \geq 5r + \epsilon$ and $N \geq N_1$. Using Lemma B.1,b) we obtain analogously

$$J_2 \leq \mathrm{const.}\big(\frac{r}{N}\big)^\alpha e^{(\omega_0+\epsilon)t}$$

for $t \geq 5r + \epsilon$ and $N \geq N_1$. Combining the estimates for J_1 and J_2 we get (6.1).

For the proof of (6.2) we first note that $S^N(t)P^N z - S(t)z = \big(\iota(L^N S^N(t)P^N z - S(t)z)(0), \dots\big)$ for $z \in \operatorname{dom}\mathcal{A}$. Therefore (6.1) implies

$$\big|\iota(L^N S^N(t)P^N z - S(t)z)(0)\big| \leq c_1\big(\frac{r}{N}\big)^\alpha e^{(\omega_0+\epsilon)t} \tag{6.4}$$

for $t \geq 5r + \epsilon$ and $N \geq N_1$. It remains to estimate

$$\begin{aligned}&\|\mathcal{A}L^N S^N(t)P^N - \mathcal{A}S(t)\|_{\mathcal{L}(Z,Z)} \\ &\qquad \leq \|\mathcal{A}^N S^N(t)P^N - P^N\mathcal{A}S(t)\|_{\mathcal{L}(Z,Z)} \\ &\qquad\quad + \|P^N\mathcal{A}S(t) - \mathcal{A}S(t)\|_{\mathcal{L}(Z,Z)}.\end{aligned} \tag{6.5}$$

From (4.10) and (4.12) we immediately get

$$\begin{aligned}&\mathcal{A}^N S^N(t)P^N - P^N\mathcal{A}S(t) \\ &\qquad = \frac{1}{2\pi i}\int_{\Gamma_{\omega_1}} \lambda e^{\lambda t}\Big((\lambda I - \mathcal{A}^N)^{-1}P^N - P^N(\lambda I - \mathcal{A})^{-1}\Big)\,d\lambda\end{aligned}$$

for $t > 3b$ and $N \geq N_1$. Estimates analogous to those leading to (6.1) (using Lemma B.1) give

$$\|\mathcal{A}^N S^N(t)P^N - P^N\mathcal{A}S(t)\|_{\mathcal{L}(Z,Z)} \leq \mathrm{const.}\big(\frac{r}{N}\big)^\alpha e^{(\omega_0+\epsilon)t} \tag{6.6}$$

for $t \geq 6r + \epsilon$ and $N \geq N_1$. Note that in the estimates we pick up an additional factor $|\lambda|$ which is bounded by "const.$|\mathrm{Im}\,\lambda|$" on $\Sigma_{a,\omega}$. We still have to estimate $\|(P^N - I)\mathcal{A}S(t)\|_{\mathcal{L}(Z,Z)}$. Let $t > 2b$. Since $S(t)$ is differentiable for $t > r$, we have $\mathcal{A}S(2b)z \in \operatorname{dom}\mathcal{A}$ and therefore

$$\begin{aligned}\|P^N\mathcal{A}S(t)z - \mathcal{A}S(t)z\| &= \|P^N S(t-2b)\mathcal{A}S(2b)z - S(t-2b)\mathcal{A}S(2b)z\| \\ &\leq \text{const.}\frac{r}{N}\|\iota S(t-2b)\mathcal{A}S(2b)z\|_{H^1}.\end{aligned} \tag{6.7}$$

Here we have used that

$$\|P^N(\varphi(0),\varphi) - (\varphi(0),\varphi)\| \leq \text{const.}\frac{r}{N}\|\varphi\|_{H^1} \quad \text{for } \varphi \in H^1,$$

(compare [19;Proposition A.3]). By differentiability of $S(t)$ we get from [20; Lemma 4.2] that $\mathcal{A}S(2b)$ and $\mathcal{A}^2S(2b)$ are bounded operators. Thus

$$\|\iota S(t-2b)\mathcal{A}S(2b)z\|_{L^2} \leq e^{(\omega_0+\epsilon)(t-2b)}\|\mathcal{A}S(2b)\|\,\|z\|$$

and

$$\|\iota\mathcal{A}S(t-2b)\mathcal{A}S(2b)z\|_{L^2} = \|\iota S(t-2b)\mathcal{A}^2S(2b)z\|_{L^2} \leq e^{(\omega_0+\epsilon)(t-2b)}\|\mathcal{A}^2S(2b)\|\,\|z\|$$

i.e., $\|\iota S(t-2b)\mathcal{A}S(2b)z\|_{H^1} \leq \text{const.}e^{(\omega_0+\epsilon)t}\|z\|$, which together with (6.7) gives

$$\|P^N\mathcal{A}S(t)z - \mathcal{A}S(t)z\| \leq \text{const.}\frac{r}{N}e^{(\omega_0+\epsilon)t}\|z\| \tag{6.8}$$

for all $t > 2b$ and $N \geq N_1$. The estimate (6.2) now follows from (6.4)-(6.6) and (6.8).

In order to prove part b) of the theorem we choose $\lambda \in \rho(\mathcal{A})$. Then Lemma B.1,b), Lemma 3.2,b) and (3.1) imply

$$\|(\lambda I - \mathcal{A}^N)^{-1}(S^N(t)P^N - P^N S(t))z\| \leq \text{const.}\left(\frac{r}{N}\right)^\alpha\|z\|_{\operatorname{dom}\mathcal{A}}$$

for all $t \in [0,T]$, $z \in \operatorname{dom}\mathcal{A}$ and N sufficiently large. This together with (3.3), Lemma B.1,b) and Lemma 3.2,b) implies

$$\begin{aligned}&\|(P^N S(t) - S^N(t)P^N)(\lambda I - \mathcal{A})^{-1}z\| \\ &\qquad \leq c\left(\frac{r}{N}\right)^\alpha\|S(t)z\| + \text{const.}\left(\frac{r}{N}\right)^\alpha\|z\|_{\operatorname{dom}\mathcal{A}} + \text{const.}\left(\frac{r}{N}\right)^\alpha\|z\|\end{aligned}$$

for $0 \leq t \leq T$, N sufficiently large and $z \in \operatorname{dom}\mathcal{A}$. The last inequality shows

$$\|P^N S(t)z - S^N(t)P^N z\| \leq \text{const.}\left(\frac{r}{N}\right)^\alpha\|z\|_{\operatorname{dom}\mathcal{A}^2}$$

for $0 \leq t \leq T$, N sufficiently large and $z \in \operatorname{dom}\mathcal{A}^2$. ∎

The rest of this section is devoted to optimal convergence rates, i.e., rate estimates of the form $O(1/N^2)$ and $O(1/N^{3/2})$, respectively.

THEOREM 6.2. *Let $\beta = 2$ if (H) is satisfied, otherwise $\beta = 3/2$.*
a) For any $\epsilon > 0$ there exist positive constants c_1, c_2 and N_1 such that for $N \geq N_1$

$$\|S^N(t)P^N z - P^N S(t)z\| \leq c_1\left(\frac{r}{N}\right)^\beta e^{(\omega_0+\epsilon)t}\|z\|_{\operatorname{dom}\mathcal{A}}, \quad t \geq 6r + \epsilon, \tag{6.9}$$

and

$$\|\iota L^N S^N(t)P^N z - P_1^N \iota S(t)z\|_{H^1} \leq c_2\left(\frac{r}{N}\right)^\beta e^{(\omega_0+\epsilon)t}\|z\|_{\operatorname{dom}\mathcal{A}}, \quad t \geq 7r + \epsilon, \tag{6.10}$$

if $z \in \operatorname{dom}\mathcal{A}$.
b) For any $\epsilon > 0$ there exist positive constants c_3, c_4 and N_2 such that for $N \geq N_2$

$$\|S^N(t)P^N z - P^N S(t)z\| \leq c_3\left(\frac{r}{N}\right)^2 e^{(\omega_0+\epsilon)t}(|\eta| + \|\varphi\|_{W^{1,\infty}}), \quad t \geq 6r + \epsilon, \tag{6.11}$$

and

$$\|\iota L^N S^N(t)P^N z - P_1^N \iota S(t)z\|_{H^1} \leq c_4\left(\frac{r}{N}\right)^2 e^{(\omega_0+\epsilon)t}(|\eta| + \|\varphi\|_{W^{1,\infty}}), \quad t \geq 7r + \epsilon, \tag{6.12}$$

for all $z = (\eta, \varphi)$ with $\varphi \in W^{1,\infty}(-r, 0; \mathbf{R}^n)$.
c) For any $T > 0$ there exsist positive constants c_5 andN_3 such that

$$\|S^N(t)P^N z - P^N S(t)z\| \leq c_5\left(\frac{r}{N}\right)^\beta \|z\|_{\operatorname{dom}\mathcal{A}^3}, \quad 0 \leq t \leq T. \tag{6.13}$$

for all $z \in \operatorname{dom}\mathcal{A}^3$ and $N \geq N_3$. For $z \in \operatorname{dom}\mathcal{A}^4$ we can replace the right-hand side of this estimate by $c_5\left(\frac{r}{N}\right)^2\|z\|_{\operatorname{dom}\mathcal{A}^4}$.

PROOF: As in the proof of Theorem 6.1 we get for N sufficiently large, $N \geq N_2$,

$$\begin{aligned}
\|P^N S(t)z - S^N(t)P^N z\| &\leq \frac{1}{2\pi}\left\|\int_{\Gamma_{1,\omega_1}\cup\Gamma_{3,\omega_1}} e^{\lambda t}\Big(P^N(\lambda I - \mathcal{A})^{-1}z - (\lambda I - \mathcal{A}^N)^{-1}P^N z\Big)\,d\lambda\right\| \\
&\quad + \frac{1}{2\pi}\left\|\int_{\Gamma_{2,\omega_1}} e^{\lambda t}\Big(P^N(\lambda I - \mathcal{A})^{-1}z - (\lambda I - \mathcal{A}^N)^{-1}P^N z\Big)\,d\lambda\right\| \\
&=: \frac{1}{2\pi}(J_1 + J_2).
\end{aligned}$$

Using the parameter representation (5.1) for Γ_{1,ω_1} and Lemma B.2,a) we get (observe also $6b \leq 6r + \epsilon$)

$$\begin{aligned}
J_1 &\leq 2c\left(\frac{r}{N}\right)^\beta \|z\|_{\operatorname{dom}\mathcal{A}}\int_{\Gamma_{1,\omega_1}} e^{t\operatorname{Re}\lambda}|\operatorname{Im}\lambda|^5|d\lambda| \\
&\leq \text{const.}\left(\frac{r}{N}\right)^\beta \|z\|_{\operatorname{dom}\mathcal{A}}\int_{-\infty}^{\omega_0+\epsilon} e^{\tau(t-6b)}d\tau \leq \text{const.}\left(\frac{r}{N}\right)^\beta e^{(\omega_0+\epsilon)t}\|z\|_{\operatorname{dom}\mathcal{A}}
\end{aligned}$$

for $t \geq 6r + \epsilon$ and $N \geq N_2$. Using Lemma B.2,b) we obtain analogously

$$J_2 \leq \text{const.}\big(\frac{r}{N}\big)^{\beta} e^{(\omega_0+\epsilon)t}\|z\|_{\text{dom}\,\mathcal{A}}$$

for $t \geq 6r + \epsilon$ and $N \geq N_2$. If $\varphi \in W^{1,\infty}(-r,0;\mathbf{R}^n)$ then in the estimates for J_1 and J_2 we can take $\beta = 2$ and $|\eta| + \|\varphi\|_{W^{1,\infty}}$ instead of $\|z\|_{\text{dom}\,\mathcal{A}}$. This finishes the proof for (6.9) and (6.11).

Since $\big(\iota L^N S^N(t)P^N z - P_1^N \iota S(t)z\big)(0)$ equals the vector component of $S^N(t)P^N z - P^N S(t)z$, we get from (6.9) the estimate

$$\Big|\Big(\iota L^N S^N(t)P^N z - P_1^N \iota S(t)z\Big)(0)\Big| \leq c_1\big(\frac{r}{N}\big)^{\beta} e^{(\omega_0+\epsilon)t}\|z\|_{\text{dom}\,\mathcal{A}}$$

(resp. $\leq c_1\big(\frac{r}{N}\big)^2 e^{(\omega_0+\epsilon)t}(|\eta| + \|\varphi\|_{W^{1,\infty}})$) for $t \geq 6r+\epsilon$ and $N \geq N_2$. It remains to estimate

$$\|\iota \mathcal{A} L^N S^N(t)P^N z - \iota \mathcal{A}\tilde{P}_1^N S(t)z\|_{L^2} \leq \|\mathcal{A}^N S^N(t)P^N z - \mathcal{A}\tilde{P}_1^N S(t)z\|.$$

where we have put $\tilde{P}_1^N = \iota^{-1}P_1^N\iota$, i.e., $\tilde{P}_1^N$ is the orthogonal projection $\text{dom}\,\mathcal{A} \to Z_1^N$. Since $\mathcal{A}^N$ and $\mathcal{A}\tilde{P}_1^N$ are bounded operators, we get from (4.9) and (4.11)

$$\begin{aligned} &\mathcal{A}^N S^N(t)P^N z - \mathcal{A}\tilde{P}_1^N S(t)z \\ &\qquad = \frac{1}{2\pi i}\int_{\Gamma_{\omega_1}} e^{\lambda t}\Big(\mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1}P^N z - \mathcal{A}\tilde{P}_1^N(\lambda I - \mathcal{A})^{-1}z\Big)\,d\lambda \end{aligned} \tag{6.14}$$

for $t > 2b$. Easy computations using also $(\mathcal{A}^N)^{-1}P^N = P^N\mathcal{A}^{-1}P^N$ and $\mathcal{A}^{-1}P^N\mathcal{A}\tilde{P}_1^N = \tilde{P}_1^N$ yield

$$\begin{aligned} &\mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1}P^N z - \mathcal{A}\tilde{P}_1^N(\lambda I - \mathcal{A})^{-1}z \\ &\qquad = \mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1}(\mathcal{A}\tilde{P}_1^N - P^N\mathcal{A})(\lambda I - \mathcal{A})^{-1}z \\ &\qquad\quad + \lambda\mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1}P^N(I - \tilde{P}_1^N)(\lambda I - \mathcal{A})^{-1}z. \end{aligned} \tag{6.15}$$

From $\mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1} = \lambda(\lambda I - \mathcal{A}^N)^{-1} - I$ and Theorem 4.1,a) we get

$$\|\mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1}P^N\| \leq \text{const.}|\text{Im}\,\lambda|^2 \quad \text{for } \lambda \in \Sigma_{a,\omega} \text{ and } N \geq N_2. \tag{6.16}$$

With the notation used in the proof of Lemma B.2 we have

$$(I - \tilde{P}_1^N)(\lambda I - \mathcal{A})^{-1}z = (\psi(0) - \psi^N(0), \psi - \psi^N) = (0, \psi - \psi^N). \tag{6.17}$$

An easy computation (see Lemma 2.1 for the definition of P^N and P_1^N) shows

$$(\mathcal{A}\tilde{P}_1^N - P^N\mathcal{A})(\psi(0), \psi) = (L(\psi^N - \psi), 0).$$

This and (6.15)-(6.17) imply

$$\begin{aligned}\|\mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1}P^N z - \mathcal{A}\tilde{P}_1^N(\lambda I - \mathcal{A})^{-1}z\| \\ \le \text{const.}|\lambda|\,|\mathrm{Im}\,\lambda|^2\big(L(\psi^N - \psi), \psi - \psi^N\big) \\ \le \text{const.}|\mathrm{Im}\,\lambda|^3\big(|L(\psi^N - \psi)| + \|\psi - \psi^N\|_{L^2}\big)\end{aligned}$$

for $\lambda \in \Sigma_{a,\omega}$ and $N \ge N_2$. The estimates used in the proof of Lemma B.2 show that

$$\begin{aligned}\|\mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1}P^N z - \mathcal{A}\tilde{P}_1^N(\lambda I - \mathcal{A})^{-1}z\| \\ \le \text{const.}|\mathrm{Im}\,\lambda|^6\big(\frac{r}{N}\big)^{\beta}\|z\|_{\mathrm{dom}\,\mathcal{A}} \quad \text{for } \lambda \in \Sigma_{a,\omega} \text{ and } N \ge N_2.\end{aligned} \tag{6.18}$$

If $\varphi \in W^{1,\infty}(-r,0;\mathbf{R}^n)$ then $\beta = 2$ and $\|z\|_{\mathrm{dom}\,\mathcal{A}}$ can be replaced by $|\eta| + \|\varphi\|_{W^{1,\infty}}$. An analogous proof shows that

$$\begin{aligned}\|\mathcal{A}^N(\lambda I - \mathcal{A}^N)^{-1}P^N z - \mathcal{A}\tilde{P}_1^N(\lambda I - \mathcal{A})^{-1}z\| \\ \le \text{const.}\big(\frac{r}{N}\big)^{\beta}\|z\|_{\mathrm{dom}\,\mathcal{A}} \quad \text{for } \lambda \in \Gamma_{2,\omega_1} \text{ and } N \ge N_2.\end{aligned} \tag{6.19}$$

Estimates (6.18) and (6.19) together with (6.15) yield

$$\|\mathcal{A}^N S^N(t)P^N z - \mathcal{A}\tilde{P}_1^N S(t)z\| \le \text{const.}\big(\frac{r}{N}\big)^{\beta}\epsilon^{(\omega_0+\epsilon)t}\|z\|_{\mathrm{dom}\,\mathcal{A}}$$

for $t \ge 7r + \epsilon$ and $N \ge N_2$ with the obvious modification in case $\varphi \in W^{1,\infty}(-r,0;\mathbf{R}^n)$.

For the proof of part c) we choose $\lambda \in \rho(\mathcal{A})$. Then Lemma B.2,b), Lemma 3.2.b) and (3.1) imply

$$\Big\|(\lambda I - \mathcal{A}^N)^{-1}\Big(S^N(t)P^N z - P^N S(t)z\Big)\Big\| \le \text{const.}\big(\frac{r}{N}\big)^{\beta}\|z\|_{\mathrm{dom}\,\mathcal{A}^2}$$

for $0 \le t \le T$, $z \in \mathrm{dom}\,\mathcal{A}^2$ and N sufficiently large. This estimate together with (3.3) and again Lemma B.2,b), Lemma 3.2,b) implies (6.13). The result for $z \in \mathrm{dom}\,\mathcal{A}^4$ is clear if we observe that $|\eta| + \|\varphi\|_{W^{1,\infty}} \le \text{const.}\|z\|_{\mathrm{dom}\,\mathcal{A}^2}$. ∎

Remarks. 1. The result of Theorem 6.2,c) shows that the vector component of $S^N(t)P^N z$ converges to the vector component of $S(t)z$ (which is the solution $x(t)$ of (1.1)) with rate $1/N^2$ uniformly on bounded t-intervals for sufficiently smooth initial data (e.g., $z \in \mathrm{dom}\,\mathcal{A}^4$). Note that the vector components of $S(t)z$ and $P^N S(t)z$ coincide.
2. The condition $z \in \mathrm{dom}\,\mathcal{A}$ for the estimates (6.9) and (6.10) can be replaced by the condition $z = (\eta, \varphi)$, $\varphi \in W^{1,2}(-r,0;\mathbf{R}^n)$, as can be seen from (B.14).
3. Since $S(t)$ is compact for $t \ge r$, the estimate (6.1) also implies

$$\|S^N(t)P^N - S(t)\|_{\mathcal{L}(Z,Z)} \to 0 \quad \text{as } N \to \infty$$

for $t \ge 5r + \epsilon$.
4. One should observe that the estimates (6.10) and (6.12) are estimates for the H^1-norm, i.e., involve the derivative of $\iota L^N S^N(t)P^N z - P_1^N \iota S(t)z$.

7. THE LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM

We consider the following control problem:

For given initial data $z = (\eta, \varphi) \in Z$ minimize the cost functional

$$J(u, z) = \int_0^\infty (|y(t)|^2 + |u(t)|^2)\, dt \tag{7.1}$$

over $u \in L^2(0, \infty; \mathbf{R}^m)$ subject to (1.1) or, equivalently, to (3.1).

DEFINITION 7.1. a) *The pair $(\mathcal{A}, \mathcal{B})$ is stabilizable if there exists an operator $\mathcal{K} \in \mathcal{L}(Z, \mathbf{R}^m)$ such that $\mathcal{A} - \mathcal{B}\mathcal{K}$ generates an exponentially stable semigroup on Z.*

b) The pair $(\mathcal{C}, \mathcal{A})$ is detectable if there exists an operator $\mathcal{G} \in \mathcal{L}(\mathbf{R}^p, Z)$ such that $\mathcal{A} - \mathcal{G}\mathcal{C}$ generates an exponentially stable semigroup on Z.

The following theorem is well-known (see [24]).

THEOREM 7.2. *Let $(\mathcal{A}, \mathcal{B})$ be stabilizable and $(\mathcal{C}, \mathcal{A})$ be detectable. Then the algebraic Riccati equation*

$$(\mathcal{A}^*\Pi + \Pi\mathcal{A} - \Pi\mathcal{B}\mathcal{B}^*\Pi + \mathcal{C}^*\mathcal{C})z = 0, \quad z \in \operatorname{dom} \mathcal{A}, \tag{7.2}$$

has a unique self-adjoint and non-negative solution Π. Moreover, the operator $\mathcal{A} - \mathcal{B}\mathcal{B}^*\Pi$ generates an exponentially stable semigroup $T(t)$, $t \geq 0$, on Z. The optimal control for (7.1) is given by

$$\hat{u}(t) = -\mathcal{B}^*\Pi T(t)z, \quad t \geq 0.$$

Remark. Without loss of generality we can assume in Definition 7.1 that $\mathcal{K}^*\mathbf{R}^m \subset \operatorname{dom} \mathcal{A}^*$ and $\mathcal{G}\mathbf{R}^p \subset \operatorname{dom} \mathcal{A}$. In fact, the solution Π of (7.2) satisfies $\Pi Z \subset \operatorname{dom} \mathcal{A}^*$ (see [7]), so that $\Pi\mathcal{B}\mathbf{R}^m \subset \operatorname{dom} \mathcal{A}^*$. Similarly the dual Riccati equation

$$(\mathcal{A}\Sigma + \Sigma\mathcal{A}^* - \Sigma\mathcal{C}^*\mathcal{C}\Sigma + \mathcal{B}\mathcal{B}^*)z = 0, \quad z \in \operatorname{dom} \mathcal{A}^*,$$

has a unique, self-adjoint and non-negative solution Σ such that $\Sigma Z \subset \operatorname{dom} \mathcal{A}$ and $\mathcal{A} - \Sigma\mathcal{C}^*\mathcal{C}$ generates an exponentially stable semigroup on Z (cf. [22]). Thus $\Sigma\mathcal{C}^*\mathbf{R}^p \subset \operatorname{dom} \mathcal{A}$.

Let $\mathcal{B}^N = P^N\mathcal{B}$ and $\mathcal{C}^N = \mathcal{C}|_{Z^N}$. The following result is concerned with stabilizability of $(\mathcal{A}^N, \mathcal{B}^N)$ and detectability of $(\mathcal{C}^N, \mathcal{A}^N)$, both uniformly with respect to N.

THEOREM 7.3. a) *Suppose that $(\mathcal{A}, \mathcal{B})$. Then there exist constants $M_1 \geq 1$, $\omega_1 > 0$ and a sequence $\mathcal{K}^N \in \mathcal{L}(Z^N, \mathbf{R}^m)$ such that $\|\mathcal{K}^N\|$ is uniformly bounded and*

$$\|e^{(\mathcal{A}^N - \mathcal{B}^N\mathcal{K}^N)t}\| \leq M_1 e^{-\omega_1 t}, \quad t \geq 0,$$

for N sufficiently large.

b) Suppose that $(\mathcal{C}, \mathcal{A})$ is detectable. Then there exist constants $M_2 \geq 1$, $\omega_2 > 0$ and a sequence $\mathcal{G}^N \in \mathcal{L}(\mathbf{R}^p, Z^N)$ such that $\|\mathcal{G}^N\|$ is uniformly bounded and

$$\|e^{(\mathcal{A}^N - \mathcal{G}^N\mathcal{C}^N)t}\| \leq M_2 e^{-\omega_2 t}, \quad t \geq 0,$$

for N sufficiently large.

PROOF: a) Let $(\mathcal{A},\mathcal{B})$ be stabilizable. Then there exists an operator $\mathcal{K} \in \mathcal{L}(Z,\mathbf{R}^m)$ such that $\mathcal{A}-\mathcal{B}\mathcal{K}$ generates an exponentially stable semigroup. We put $\mathcal{K}^N = \mathcal{K}|_{Z^N}$, $N = 1,2,\dots$. Then, for $z \in Z^N$,

$$\begin{aligned}(\mathcal{A}^N - \mathcal{B}^N\mathcal{K}^N)z &= \mathcal{A}^N z - \mathcal{B}\mathcal{K}z = \mathcal{A}L^N z - \mathcal{B}\mathcal{K}P^N L^N z\\ &= (\mathcal{A}-\mathcal{B}\mathcal{K}P^N)L^N z,\end{aligned}$$

i.e., the semigroup $e^{(\mathcal{A}^N-\mathcal{B}^N\mathcal{K}^N)t}$ is the approximating semigroup for the delay system

$$\dot{x}(t) = \hat{L}_N(x_t),$$

which corresponds to $\mathcal{A}-\mathcal{B}\mathcal{K}P^N$. Let $\mathcal{K}(\eta,\varphi) = K_0\eta + \int_{-r}^0 K(\theta)\varphi(\theta)\,d\theta$ for $(\eta,\varphi)\in Z$, where $K_0 \in \mathbf{R}^{m\times n}$ and $K(\cdot) \in L^2(-r,0;\mathbf{R}^{m\times n})$. An easy computation shows that

$$\mathcal{K}P^N(\eta,\varphi) = K_0\eta + \int_{-r}^0 K^N(\theta)\varphi(\theta)\,d\theta,$$

where $K^N = \sum_{j=1}^N K_j^N E_j^N$ with $K_j^N = (N/r)\int_{t_j^N}^{t_{j-1}^N} K(\theta)\,d\theta$. Therefore

$$\hat{L}_N(\varphi) = L(\varphi) - BK_0\varphi(0) - B\int_{-r}^0 K^N(\theta)\varphi(\theta)\,d\theta.$$

The delay system corresponding to $\mathcal{A}-\mathcal{B}\mathcal{K}$ is $\dot{x}(t) = \hat{L}(x_t)$, where

$$\hat{L}(\varphi) = L(\varphi) - BK_0\varphi(0) - B\int_{-r}^0 K(\theta)\varphi(\theta)\,d\theta.$$

We choose γ as in Lemma 2.4 corresponding to $\hat{L}$ and $\omega > \gamma$, $b > r$, $0 < \delta < 1$ as in Section 4. The estimate

$$\begin{aligned}|\hat{L}_N(e_\lambda^N I_n)| &\le |\hat{L}(e_\lambda^N I_n)| + |\hat{L}_N(e_\lambda^N I_n) - \hat{L}(e_\lambda^N I_n)|\\ &\le (1-\delta)|\lambda||B|\int_{-r}^0 |K(\theta)-K^N(\theta)|\,|e_\lambda^N(\theta)|\,d\theta\\ &\le (1-\delta)|\lambda| + |B|r^{1/2}\|K-K^N\|_{L^2}\max_{0\le k\le N}|\mu(\lambda r/N)|^k\\ &\le (1-\delta)|\lambda| + |B|r^{1/2}\|K-K^N\|_{L^2}K_0e^{-ab}|\mathrm{Im}\,\lambda|\end{aligned}$$

for $\lambda \in \Sigma_{a,\omega}$ shows that for $\hat{\delta} \in (0,\delta)$ there exists an N_0 such that

$$|\hat{L}_N(e_\lambda^N I_n)| \le (1-\hat{\delta})|\lambda| \quad \text{for } \lambda \in \Sigma_{a,\omega} \text{ and } N \ge N_0.$$

Using this estimate we can establish Theorems 4.1 and 4.2 for the operators $\mathcal{A}^N - \mathcal{B}^N\mathcal{K}^N$, $N \geq N_0$. But then also Theorem 5.1 is valid for $\mathcal{A} - \mathcal{B}\mathcal{K}$ and $\mathcal{A}^N - \mathcal{B}^N\mathcal{K}^N$, $N \geq N_0$, which proves the result on stabilizability.

b) Let $(\mathcal{C}, \mathcal{A})$ be detectable. Then there exists an operator $\mathcal{G} \in \mathcal{L}(\mathbf{R}^p, Z)$ with $\mathcal{G}\mathbf{R}^p \subset \operatorname{dom}\mathcal{A}$ such that $\mathcal{A} - \mathcal{G}\mathcal{C}$ generates an exponentially stable semigroup $T(t)$, $t \geq 0$, on Z. Since $T(t)$ is not the solution semigroup of a delay system the proof in this case will be somewhat different compared to that for a).

For $N = 1, 2, \ldots$ we define

$$\mathcal{G}^N = P^N \tilde{P}_1^N \mathcal{G} \in \mathcal{L}(\mathbf{R}^p, Z^N).$$

Since $\mathcal{G}y \in \operatorname{dom}\mathcal{A}$ for all $y \in \mathbf{R}^p$, there exists a function $G \in H^1(-r, 0; \mathbf{R}^{n \times p})$ such that

$$\mathcal{G}y = (G(0)y, Gy) \quad \text{for all } y \in \mathbf{R}^p. \tag{7.3}$$

Using the definition of $\mathcal{G}^N$ and Lemma 2.1 we get

$$\mathcal{G}^N y = (G(0)y, \tilde{G}^N y) \quad \text{for all } y \in \mathbf{R}^p, \tag{7.4}$$

where $\tilde{G}^N = (1/2)\sum_{k=1}^N E_k^N (G(t_k^N) + G(t_{k-1}^N))$. The matrix representation G^N of $\mathcal{G}^N$ is given by

$$G^N = \operatorname{col}(G_0^N, \ldots, G_N^N) \in \mathbf{R}^{n(N+1) \times p},$$

where $G_0^N = G(0)$ and $G_k^N = (1/2)(G(t_{k-1}^N) + G(t_k^N))$, $k = 1, \ldots, N$. From (7.3) and (7.4) it is easily seen that

$$\|\mathcal{G}^N - \mathcal{G}\|^2 \leq \frac{1}{4}\sum_{k=1}^N \int_{t_k^N}^{t_{k-1}^N} \Big(|G(t_k^N) - G(\theta)| + |G(t_{k-1}^N) - G(\theta)|\Big)^2 d\theta \leq r\omega\left(\frac{r}{N}; G\right)^2$$

where $\omega(\epsilon; G) = \sup_{|\theta - \tau| \leq \epsilon} |G(\theta) - G(\tau)|$. Therefore

$$\lim_{N \to \infty} \|\mathcal{G}^N - \mathcal{G}\| = 0 \tag{7.5}$$

From $\mathcal{A}\mathcal{G}y = \big(L(Gy), \dot{G}y\big)$ and

$$\mathcal{A}^N\mathcal{G}^N y = \mathcal{A}\tilde{P}_1^N \mathcal{G}y = \left(L\Big(\sum_{k=1}^N B_k^N G(t_k^N) y\Big), \frac{N}{r}\sum_{k=1}^N E_k^N (G(t_{k-1}^N) - G(t_k^N))y\right)$$

it follows immediately that (because $\sum_{k=0}^N B_k^N G(t_k^N) = \tilde{P}_1^N G$ and $\tilde{P}_1^N G \to G$ with respect to the graph norm)

$$\lim_{N \to \infty} \|\mathcal{A}^N\mathcal{G}^N - \mathcal{A}\mathcal{G}\| = 0. \tag{7.6}$$

The representation (4.10) together with Theorem 4.1,a) shows that for $t_0 = 4r$

$$\|\mathcal{A}^N S^N(t_0)\| \le M \quad \text{for all } N, \tag{7.7}$$

where the constant M does not depend on N.

Let $T^N(t)$, $t \ge 0$, denote the semigroup generated by $\mathcal{A}^N - \mathcal{G}^N\mathcal{C}^N$. By the variation of constants formula we have

$$T^N(t) = S^N(t) - \int_0^t S^N(t-s)\mathcal{G}^N\mathcal{C}^N T^N(s)\,ds, \quad t \ge 0. \tag{7.8}$$

By Gronwall's inequality and (7.5) we obtain

$$\|T^N(t)z\| \le \text{const.}\|z\|, \quad 0 \le t \le t_0, \; N = 1,2,\dots\,. \tag{7.9}$$

This together with (7.6) and (7.7) gives

$$\|\mathcal{A}^N T^N(t_0)z\| \le M\|z\| + \text{const.}\|\mathcal{A}^N\mathcal{G}^N\|\,\|z\| \le \tilde{M}\|z\|,$$

for all N, where $\tilde{M}$ is not dependent on N. From [20; p.56], we see

$$T^N(t) = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}\Big(\lambda I - (\mathcal{A}^N - \mathcal{G}^N\mathcal{C}^N)\Big)^{-1} d\lambda, \quad t > 2t_0, \tag{7.10}$$

where $\tilde{\Gamma} = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_3$ and

$$\begin{aligned}
\tilde{\Gamma}_1 &= \big\{\lambda \mid \operatorname{Im}\lambda = e^{(\alpha - \operatorname{Re}\lambda)t_0},\ \operatorname{Re}\lambda \le \beta\big\}, \\
\tilde{\Gamma}_2 &= \big\{\lambda \mid |\operatorname{Im}\lambda| \le e^{(\alpha-\beta)t_0},\ \operatorname{Re}\lambda = \beta\big\}, \\
\tilde{\Gamma}_3 &= \big\{\lambda \mid \bar{\lambda} \in \tilde{\Gamma}_1\big\},
\end{aligned}$$

with $\alpha = (1/t_0)\ln 2\tilde{M}$, $\beta = \gamma + \sup_N \|\mathcal{G}^N\mathcal{C}^N\|$ (γ the constant defined in Lemma 2.4). Moreover (see [20; Thm. 4.4]) we have

$$\Big\|\Big(\lambda I - (\mathcal{A}^N - \mathcal{G}^N\mathcal{C}^N)\Big)^{-1}\Big\| \le \text{const.}|\operatorname{Im}\lambda| \tag{7.11}$$

for $\lambda \in \Sigma_{\alpha,\beta} = \big\{\lambda \mid |\operatorname{Im}\lambda| \ge e^{(\alpha - \operatorname{Re}\lambda)t_0},\ \operatorname{Re}\lambda \le \beta\big\}$, where "const." does not depend on N.

For $z \in Z$ we consider the equation

$$\Big(\lambda I - (\mathcal{A}^N - \mathcal{G}^N\mathcal{C}^N)\Big)z^N = P^N z. \tag{7.12}$$

Let $\lambda_0 \in \rho(\mathcal{A})$ and put

$$\mathcal{F}_\lambda = I + (\lambda_0 I - \mathcal{A})^{-1}\big((\lambda - \lambda_0)I + \mathcal{G}\mathcal{C}\big).$$

It is easy to see that $\lambda \in \rho(\mathcal{A}-\mathcal{G}\mathcal{C})$ if and only if $\mathcal{F}_\lambda$ is continuously invertible. If this is the case then

$$(7.13)\qquad \Big(\lambda I-(\mathcal{A}-\mathcal{G}\mathcal{C})\Big)^{-1}=\mathcal{F}_\lambda^{-1}(\lambda_0 I-\mathcal{A})^{-1}.$$

Since $\lambda_0 \in \rho(\mathcal{A}^N)$ for N sufficiently large, we analogously have

$$(7.14)\qquad \Big(\lambda I-(\mathcal{A}^N-\mathcal{G}^N\mathcal{C}^N)\Big)^{-1}=(\mathcal{F}_\lambda^N)^{-1}(\lambda_0 I-\mathcal{A}^N)^{-1}$$

for $\lambda \in \rho(\mathcal{A}^N-\mathcal{G}^N\mathcal{C}^N)$ and N sufficiently large, where

$$\mathcal{F}_\lambda^N=I+(\lambda_0 I-\mathcal{A}^N)^{-1}\Big((\lambda-\lambda_0)I+\mathcal{G}^N\mathcal{C}^N\Big).$$

Using $I=\mathcal{F}_\lambda-(\lambda_0 I-\mathcal{A})^{-1}((\lambda-\lambda_0)I+\mathcal{G}\mathcal{C})$, $\lambda\in\rho(\mathcal{A}-\mathcal{G}\mathcal{C})$, we see that equation (7.14) is equivalent to

$$(7.15)\qquad \mathcal{F}_\lambda(I+\mathcal{E}_\lambda^N)z^N=(\lambda_0 I-\mathcal{A}^N)^{-1}P^N z$$

where

$$(7.16)\qquad \begin{aligned}\mathcal{E}_\lambda^N=\mathcal{F}_\lambda^{-1}\Big(&(\lambda-\lambda_0)((\lambda_0 I-\mathcal{A}^N)^{-1}P^N-(\lambda_0 I-\mathcal{A})^{-1})\\ &+(\lambda_0 I-\mathcal{A}^N)^{-1}\mathcal{G}^N\mathcal{C}^N-(\lambda_0 I-\mathcal{A})^{-1}\mathcal{G}\mathcal{C}\Big).\end{aligned}$$

Since $(\lambda_0 I-\mathcal{A})^{-1}$ is compact, we have

$$\|P^N(\lambda_0 I-\mathcal{A})^{-1}-(\lambda_0 I-\mathcal{A})^{-1}\|\to 0 \quad \text{as } N\to\infty$$

and therefore by Lemma B.1 also $\|(\lambda_0 I-\mathcal{A}^N)^{-1}P^N-(\lambda_0 I-\mathcal{A})^{-1}\|\to 0$. Using this and (7.5) we obtain from (7.16) that

$$\|\mathcal{E}_\lambda^N\|\to 0 \quad \text{as } N\to\infty$$

uniformly for λ in compact subsets of $\rho(\mathcal{A}-\mathcal{G}\mathcal{C})$. Therefore for any compact subset K of $\rho(\mathcal{A}-\mathcal{G}\mathcal{C})$ there exists an N_0 such that $\big(\mathcal{F}_\lambda(I+\mathcal{E}_\lambda^N)\big)^{-1}$ exists for $\lambda\in K$ and $N\ge N_0$, i.e., $K\subset\rho(\mathcal{A}^N-\mathcal{G}^N\mathcal{C}^N)$ for $N\ge N_0$. Equations (7.14) and (7.15) imply

$$(\mathcal{F}_\lambda^N)^{-1}=\sum_{k=0}^{\infty}(\mathcal{E}_\lambda^N)^k\mathcal{F}_\lambda^{-1} \quad \text{for } \lambda\in K,\ N\ge N_0.$$

This representation shows that

$$\lim_{N\to\infty}\|(\mathcal{F}_\lambda^N)^{-1}-\mathcal{F}_\lambda^{-1}\|=0$$

uniformly on compact subsets of $\rho(\mathcal{A} - \mathcal{G}\mathcal{C})$ which immediately implies

$$\lim_{N\to\infty} \|(\lambda I - (\mathcal{A}^N - \mathcal{G}^N\mathcal{C}^N))^{-1} P^N z - (\lambda I - (\mathcal{A} - \mathcal{G}\mathcal{C}))^{-1} z\| = 0 \tag{7.17}$$

uniformly on compact subsets of $\rho(\mathcal{A}-\mathcal{G}\mathcal{C})$. The last result implies (by the same arguments as those given in the proof of Theorem 5.1) that we can replace β in the definition of $\tilde{\Gamma}$ by any $\omega > \omega_2 = \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathcal{A} - \mathcal{G}\mathcal{C})\}$ and still have the representation (7.10) for N sufficiently large. As in the proof of Theorem 5.1 (but using (7.11) and (7.17) instead of Theorem 4.1) we obtain

$$\|T^N(t)\| \le \text{const.}e^{\omega t}, \quad t > 2t_0,$$

which together with (7.9) (for $t \in [0, 3t_0]$) finally gives

$$\|T^N(t)\| \le \text{const.}e^{\omega t}, \quad t \ge 0, \tag{7.18}$$

for N sufficiently large. ∎

Corresponding to problem (7.1) we consider the following approximating problems:

Given $z \in Z$ minimize

$$J^N(u; z) = \int_0^\infty \left(|\mathcal{C}^N z^N(t)|^2 + |u(t)|^2\right) dt \tag{7.19}$$

subject to

$$z^N(t) = S^N(t)P^N z + \int_0^t S^N(t-s)\mathcal{B}u(s)\, ds, \quad t \ge 0. \tag{7.20}$$

THEOREM 7.4. *Suppose that $(\mathcal{A}, \mathcal{B})$ is stabilizable and $(\mathcal{C}, \mathcal{A})$ is detectable. Then for N sufficiently large the algebraic Riccati equation associated with (7.19),*

$$(\mathcal{A}^N)^*\Pi + \Pi\mathcal{A}^N - \Pi\mathcal{B}^N(\mathcal{B}^N)^*\Pi + (\mathcal{C}^N)^*\mathcal{C}^N = 0, \tag{7.21}$$

has a unique self-adjoint, non-negative solution Π^N. There exist positive constants M_1, M_2 and ω such that

$$\|\Pi^N\| \le M_1 \quad \text{and} \quad \|e^{(\mathcal{A}^N - \mathcal{B}^N(\mathcal{B}^N)^*\Pi^N)t}\| \le M_2 e^{-\omega t}, \ t \ge 0,$$

for N sufficiently large.

PROOF: The result on existence and uniqueness of a solution Π^N of (7.21) for N sufficiently large follows immediately from Theorem 7.3 and Theorem 7.2 applied for $\mathcal{A}^N$, $\mathcal{B}^N$ and $\mathcal{C}^N$.

Let $\mathcal{K}^N \in \mathcal{L}(Z^N, \mathbf{R}^m)$ and $\mathcal{G}^N \in \mathcal{L}(\mathbf{R}^p, Z^N)$ be defined as in the proof of Theorem 7.3. Then for all $z \in Z^N$

$$\begin{aligned}\langle \Pi^N z, z\rangle &= \inf_{u \in L^2} J^N(u, z) \le J^N(-\mathcal{K}^N z^N(\cdot), z) \\ &= \int_0^\infty \left(\left|\mathcal{C}^N e^{(\mathcal{A}^N - \mathcal{B}^N\mathcal{K}^N)t} z\right|^2 + \left|\mathcal{K}^N e^{(\mathcal{A}^N - \mathcal{B}^N\mathcal{K}^N)t} z\right|^2\right) dt \le \kappa\|z\|^2.\end{aligned}$$

where κ is not dependent on N. Since Π^N is non-negative, this implies

$$\|\Pi^N\| \le \kappa \quad \text{for all } N.$$

From equation (7.21) we immediately get

$$(\mathcal{A}^N - \mathcal{B}^N(\mathcal{B}^N)^*\Pi^N)^*\Pi^N + \Pi^N(\mathcal{A}^N - \mathcal{B}^N(\mathcal{B}^N)^*\Pi^N) + \Pi^N\mathcal{B}^N(\mathcal{B}^N)^*\Pi^N + (\mathcal{C}^N)^*\mathcal{C}^N = 0.$$

Let $z^N(t) = \exp((\mathcal{A}^N - \mathcal{B}^N(\mathcal{B}^N)^*\Pi^N)t)z$, $t \ge 0$. Then a short calculation gives

$$\frac{d}{dt}\langle z^N(t), \Pi^N z^N(t)\rangle + \left|(\mathcal{B}^N)^*\Pi^N z^N(t)\right|^2 + \left|\mathcal{C}^N z^N(t)\right|^2 = 0, \quad t \ge 0,$$

or

$$\langle \Pi^N z^N(t), z^N(t)\rangle + \int_0^t \left(\left|(\mathcal{B}^N)^*\Pi^N z^N(s)\right|^2 + \left|\mathcal{C}^N z^N(s)\right|^2\right) ds = \langle \Pi^N z, z\rangle \le \kappa\|z\|^2, \quad t \ge 0.$$

As a consequence

$$\int_0^\infty \left(\left|(\mathcal{B}^N)^*\Pi^N z^N(s)\right|^2 + \left|\mathcal{C}^N z^N(s)\right|^2\right) ds \le \kappa\|z\|^2, \quad t \ge 0. \tag{7.22}$$

By the variation of constants formula we get

$$z^N(t) = T^N(t)z + \int_0^t T^N(t-s)(\mathcal{G}^N\mathcal{C}^N - \mathcal{B}^N(\mathcal{B}^N)^*\Pi^N)z^N(s)\,ds, \quad t \ge 0,$$

where $T^N(t) = \exp((\mathcal{A}^N - \mathcal{G}^N\mathcal{C}^N)t)$, $t \ge 0$. Taking norms and observing (7.5), (7.18) and (7.22) we have

$$\begin{aligned}
\|z^N(t)\| &\le Me^{\omega t}\Big(\|z\| + \sup_N\|\mathcal{G}^N\| + \int_0^t |\mathcal{C}^N z^N(s)|\,ds \\
&\qquad + \|\mathcal{B}\|\int_0^t \left|(\mathcal{B}^N)^*\Pi^N z^N(s)\right| ds\Big) \\
&\le Me^{\omega t}\Big(\|z\| + \sup_N\|\mathcal{G}^N\| t^{1/2}\big(\int_0^\infty \left|\mathcal{C}^N z^N(s)\right|^2 ds\big)^{1/2} \\
&\qquad + \|\mathcal{B}\| t^{1/2}\big(\int_0^t \left|(\mathcal{B}^N)^*\Pi^N z^N(s)\right|^2 ds\big)^{1/2}\Big) \\
&\le Me^{\omega t}\Big(1 + t^{1/2}(\sup_N\|\mathcal{G}^N\| + \kappa^{1/2}\|\mathcal{B}\|)\Big)\|z\|
\end{aligned}$$

for all $t \ge 0$ and all N. Finally we get

$$\int_0^\infty \|z^N(t)\|^2 dt \le \text{const.}\|z\|, \quad z \in Z^N,$$

where "const." is not dependent on N. Now the result follows from a theorem by Datko (see the version given in [22;Thm. 6.2]). ∎

The following theorem is an immediate consequence of Theorem 7.4 and results given in [7;Thm. 6.9] and [4;Thm. 2.2]. These results assume that the adjoint semigroups converge strongly as is the case for our scheme.

Theorem 7.5. *Suppose that $(\mathcal{A}, \mathcal{B})$ is stabilizable and $(\mathcal{C}, \mathcal{A})$ is detectable. Then the unique solution Π^N of (7.21) converges strongly to the unique solution Π of (7.2). The semigroups generated by $\mathcal{A} - \mathcal{B}\mathcal{B}^*\Pi^N P^N$ are exponentially stable uniformly with respect to $N \geq N_0$ for some $N_0 > 0$. Moreover, for any $\epsilon > 0$ there exists an $N(\epsilon)$ such that*

$$J(-\mathcal{B}^*\Pi^N P^N z(\cdot), z) \leq J(u^0, z) + \epsilon\|z\|^2$$

for all $z \in Z$ and $N \geq N(\epsilon)$, where $u^0 = -\mathcal{B}^*\Pi z(\cdot)$.

Appendix A

In this appendix we state and prove some estimates involving the Padé-approximant $\frac{1-\tau/2}{1+\tau/2}$.

LEMMA A.1. *For $m \geq 1$ and $|z| < 2m$*

$$\left|e^{-z} - \left(\frac{1-z/2m}{1+z/2m}\right)^m\right| \leq \begin{cases} \frac{1}{8}\left(\frac{|z|}{m}\right)^2\left(1+\frac{|z|}{2m}\right)e^{-\mathrm{Re}\, z}\left(\left(\frac{1+|z|/2m}{1-|z|/2m}\right)^m - 1\right) & \text{for } \mathrm{Re}\, z < 0, \\ \frac{1}{8m^2}|z|^3\frac{1+|z|/2m}{1-|z|/2m} & \text{for } \mathrm{Re}\, z \geq 0. \end{cases}$$

PROOF: Let $\rho = z/m$. Then

$$\begin{aligned} e^{-z} - \left(\frac{1-z/2m}{1+z/2m}\right)^m &= (e^{-\rho})^m - \left(\frac{1-\rho/2}{1+\rho/2}\right)^m \\ &= \left(e^{-\rho} - \frac{1-\rho/2}{1+\rho/2}\right)\sum_{k=0}^{m-1} e^{-k\rho}\left(\frac{1-\rho/2}{1+\rho/2}\right)^{m-k-1}. \end{aligned} \tag{A.1}$$

For $|\rho| < 2$

$$e^{-\rho} - \frac{1-\rho/2}{1+\rho/2} = \sum_{k=3}^{\infty}\left(\frac{1}{k!} - \left(\frac{1}{2}\right)^{k-1}\right)(-\rho)^k$$

and therefore

$$\begin{aligned} \left|e^{-\rho} - \frac{1-\rho/2}{1+\rho/2}\right| &\leq \frac{1}{12}|\rho|^3 + \sum_{k=4}^{\infty}\left(\frac{1}{2}\right)^{k-1}|\rho|^k \\ &\leq \frac{1}{8}|\rho|^3\left(1 + \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}|\rho|^k\right) = \frac{1}{8}|\rho|^3\frac{1+|\rho|/2}{1-|\rho|/2}. \end{aligned} \tag{A.2}$$

For $\mathrm{Re}\,\rho < 0$ we have

$$\begin{aligned} \left|\sum_{k=0}^{m-1} e^{-k\rho}\left(\frac{1-\rho/2}{1+\rho/2}\right)^{m-k-1}\right| &\leq \sum_{k=0}^{m-1} e^{-k\mathrm{Re}\,\rho}\left(\frac{1+|\rho|/2}{1-|\rho|/2}\right)^{m-k-1} \\ &\leq e^{-m\mathrm{Re}\,\rho}\left(\left(\frac{1+|\rho|/2}{1-|\rho|/2}\right)^m - 1\right)\left(\frac{1+|\rho|/2}{1-|\rho|/2} - 1\right)^{-1}. \end{aligned}$$

This together with (A.1) and (A.2) gives

$$\begin{aligned} \left|e^{-z} - \left(\frac{1-z/2m}{1+z/2m}\right)^m\right| &\leq \frac{1}{8}\left(\frac{|z|}{m}\right)^3\frac{1+|z|/2m}{1-|z|/2m}e^{-\mathrm{Re}\, z}\frac{1-|z|/2m}{|z|/m}\left(\left(\frac{1+|z|/2m}{1-|z|/2m}\right)^m - 1\right) \\ &= \frac{1}{8}\left(\frac{|z|}{m}\right)^2\left(1+\frac{|z|}{2m}\right)e^{-\mathrm{Re}\, z}\left(\left(\frac{1+|z|/2m}{1-|z|/2m}\right)^m - 1\right) \end{aligned}$$

for $|z| < 2m$ and $\operatorname{Re} z < 0$. In case $\operatorname{Re}\rho \geq 0$ we get

$$\Big|\sum_{k=0}^{m-1} e^{-k\rho}\Big(\frac{1-\rho/2}{1+\rho/2}\Big)^{m-k-1}\Big| \leq \sum_{k=0}^{m-1} e^{-k\operatorname{Re}\rho} = \frac{1-\epsilon^{-m\operatorname{Re}\rho}}{1-\epsilon^{-\operatorname{Re}\rho}} \leq m.$$

Therefore using again (A.1) and (A.2) we get

$$\Big|e^{-z} - \Big(\frac{1-z/2m}{1+z/2m}\Big)^m\Big| \leq \frac{1}{8m^2}|z|^3\frac{1+|z|/2m}{1-|z|/2m}$$

for $|z| < 2m$ and $\operatorname{Re} z \geq 0$. ∎

LEMMA A.2. *Let* $\alpha_k = \Big|e^{\lambda t_k^N} - \Big(\frac{1-r\lambda/2N}{1+r\lambda/2N}\Big)^k\Big|$. *Then for* $|\lambda| < \rho$ *and* N *sufficiently large* $(N > r\rho/2)$

$$\alpha_k \leq \begin{cases} \frac{1}{8N^2}\rho^3 r^3 \frac{1+\rho r/2N}{1-\rho r/2N} & \text{for } \operatorname{Re}\lambda \geq 0, \\ \frac{1}{8N^2}\rho^2 r^2 (1+\frac{\rho r}{2N})\epsilon^{r\rho}\Big(\Big(\frac{1+\rho r/2N}{1-\rho r/2N}\Big)^N - 1\Big) & \text{for } \operatorname{Re}\lambda < 0, \end{cases}$$

$k = 1, \ldots, N$.

PROOF: Let $\zeta_{N,k} = \lambda k r/N$. Then

$$\alpha_k = \Big|\epsilon^{-\zeta_{N,k}} - \Big(\frac{1-\zeta_{N,k}/2k}{1+\zeta_{N,k}/2k}\Big)^k\Big|, \quad k = 1, \ldots, N.$$

For $|\lambda| \leq \rho$ we have $|\zeta_{N,k}| \leq \rho k r/N$ and $|\zeta_{N,k}| < 2k$ for $N > r\rho/2$. Lemma A.1 for $z = \zeta_{N,k}$ implies

$$\alpha_k \leq \begin{cases} \frac{1}{8}\frac{\rho^2 r^2}{N^2}(1+\frac{\rho r}{2N})\epsilon^{kr\rho/N}\Big(\Big(\frac{1+\rho r/2N}{1-\rho r/2N}\Big)^k - 1\Big) & \text{for } \operatorname{Re}\lambda < 0, \\ \frac{1}{8}\frac{\rho^3 r^3 k}{N^3}\frac{1+\rho r/2N}{1-\rho r/2N} & \text{for } \operatorname{Re}\lambda \geq 0, \end{cases}$$

for $|\lambda| < \rho$ and $N > r\rho/2$. The estimates of the lemma are an immediate consequence. ∎

COROLLARY A.3. *For any compact set* $K \subset \mathbf{C}$ *there exists a constant* $c = c(K)$ *such that*

$$\sup_{-r\leq\theta\leq 0}\big|e_\lambda^N(\theta) - \epsilon^{\lambda\theta}\big| \leq c(K)\Big(\frac{r}{N}\Big)^2 \quad \textit{for all } N.$$

PROOF: Since $\sum_{k=0}^N \epsilon^{\lambda t_k^N} B_k^N$ is the first order interpolating spline for $\epsilon^{\lambda\theta}$, we have (see [23; Thm. 2.6])

$$\sup_{-r\leq\theta\leq 0}\Big|\epsilon^{\lambda\theta} - \sum_{k=0}^N \epsilon^{\lambda t_k^N} B_k^N(\theta)\Big| \leq \text{const.}\Big(\frac{r}{N}\Big)^2 \sup_{-r\leq\theta\leq 0}\Big|\frac{d^2}{d\lambda^2}\epsilon^{\lambda\theta}\Big|.$$

Then the result follows from the estimate

$$\begin{aligned} &\sup_{-r\leq\theta\leq 0}\Big|\sum_{k=0}^N\Big(\epsilon^{\lambda t_k^N} - \Big(\frac{1-r\lambda/2N}{1+r\lambda/2N}\Big)^k\Big)B_k^N\Big| \\ &\qquad \leq \max_{k=1,\ldots,N}\Big|\epsilon^{\lambda t_k^N} - \Big(\frac{1-r\lambda/2N}{1+r\lambda/2N}\Big)^k\Big| \leq \text{const.}\Big(\frac{r}{N}\Big)^2 \end{aligned}$$

for all $\lambda \in K$ and $N = 1, 2, \ldots$, which is an immediate consequence of Lemma A.2. ∎

COROLLARY A.4. *For any compact set $K \subset \mathbf{C}$ there exists a constant $c = c(K)$ such that*

$$|\Delta^N(\lambda) - \Delta(\lambda)| \le c(K)\Big(\frac{r}{N}\Big)^2 \quad \text{for } \lambda \in K \text{ and } N = 1, 2, \dots .$$

PROOF: By definition of $\Delta^N(\lambda)$ and $\Delta(\lambda)$ we have

$$|\Delta^N(\lambda) - \Delta(\lambda)| \le \gamma \sup_{-r \le \theta \le 0} |e_\lambda^N(\theta) - e^{\lambda\theta}|,$$

where γ is the constant defined in Lemma 2.4. Then the result follows immediately from Corollary A.3. ∎

LEMMA A.5. *There exist positive constants K_0 and a_0 such that for $0 \le j \le N$, $N = 1, 2, \dots$, and $\lambda \in \Sigma_{a,\omega}$*

$$\left|\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right|^j \le K_0 e^{-ab}|\mathrm{Im}\,\lambda|$$

provided $a \ge a_0$. In fact $K_0 = e^{b\omega}$ and $a_0 = \frac{1}{b}\big(\ln \frac{r}{b(b-r)} - 1\big)$ is a possible choice for K_0 and a_0.

PROOF: Let $\alpha_0 = \frac{2N}{r}(1 - \frac{r}{b})^{1/2}$. We consider the following three cases separately.
Case 1: $0 \le \mathrm{Re}\,\lambda \le \omega$, $\lambda \in \Sigma_{a,\omega}$.

In this case the result follows immediately from

$$\left|\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right| \le 1 \le e^{-ab} e^{b\mathrm{Re}\,\lambda}|\mathrm{Im}\,\lambda| \le e^{b\omega} e^{-ab}|\mathrm{Im}\,\lambda|.$$

Case 2: $-\alpha_0 \le \mathrm{Re}\,\lambda \le 0$, $\lambda \in \Sigma_{a,\omega}$.

We put $\lambda = \alpha + i\beta$. The inequality $\alpha \ge -\alpha_0$ implies $1 + (r/2N)\alpha > 0$ and therefore

$$\left|\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right| = \left(\frac{(1 - r\alpha/2N)^2 + (r\beta/2N)^2}{(1 + r\alpha/2N)^2 + (r\beta/2N)^2}\right)^{1/2} \le \frac{1 - r\alpha/2N}{1 + r\alpha/2N}.$$

For the function $f(\alpha) = \big(\frac{1 - r\alpha/2N}{1 + r\alpha/2N}\big)^N e^{\alpha b}$, $-2N/r < \alpha \le 0$, we have $f(0) = 1$ and $f'(\alpha) \ge 0$ for $-\alpha_0 \le \alpha \le 0$. Therefore $f(\alpha) \le 1$ on $-\alpha_0 \le \alpha \le 0$ and

$$\left|\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right|^j \le \left|\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right|^N \le e^{-b\mathrm{Re}\,\lambda} \le e^{-ab}|\mathrm{Im}\,\lambda|, \quad j = 1, \dots, N.$$

Case 3: $\mathrm{Re}\,\lambda \le -\alpha_0$, $\lambda \in \Sigma_{a,\omega}$.

Let $\lambda = -\alpha + i\beta$, $\alpha \ge \alpha_0$. Then

$$\begin{aligned}\left|\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right|^N &= \left|1 - \frac{2}{1 + r\lambda/2N}\right|^N \le \left(1 + \frac{2}{|1 + r\lambda/2N|}\right)^N \\ &\le \left(1 + \frac{2}{r|\beta|/2N}\right)^N = \left(1 + \frac{1}{N}\frac{4N^2}{r|\beta|}\right)^N.\end{aligned} \tag{A.3}$$

We claim that

$$\frac{4N^2}{r|\beta|} \leq b\alpha \quad \text{for } \lambda \in \Sigma_{a,\omega},\ \alpha \geq \alpha_0,\ N = 1, 2, \ldots, \tag{A.4}$$

provided a is sufficiently large. Then we get from (A.3)

$$\left|\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right|^j \leq \left|\frac{1 - r\lambda/2N}{1 + r\lambda/2N}\right|^N \leq \left(1 + \frac{b\alpha}{N}\right)^N \leq e^{b\alpha} \leq e^{-ab}|\mathrm{Im}\,\lambda|$$

for $j = 1, \ldots, N$ and a sufficiently large.

We now prove (A.4). Since $|\beta| \geq e^{b(a+\alpha)}$, the estimate (A.4) is certainly satisfied if $4N^2/r \leq b\alpha e^{b\alpha} e^{ab}$, $\alpha \geq \alpha_0$. Since the function xe^x is increasing on $x \geq 0$, (A.4) is satisfied if

$$\frac{4N^2}{r} \leq b\alpha_0 e^{b\alpha_0} e^{ab}. \tag{A.5}$$

The definition of α_0 shows that (A.5) is equivalent to

$$e^{ab} \geq \frac{r}{b(b-r)} \frac{2Nb}{r}\left(1 - \frac{r}{b}\right)^{1/2} \exp\left(-\frac{2Nb}{r}\left(1 - \frac{r}{b}\right)^{1/2}\right).$$

Using $xe^{-x} \leq e^{-1}$ for $x \geq 0$, we see that the last inequality is satisfied for all N provided $e^{ab} \geq r\big(b(b-r)e\big)^{-1}$, i.e.,

$$a \geq \frac{1}{b}\left(\ln \frac{r}{b(b-r)} - 1\right). \quad \blacksquare$$

Appendix B

In this appendix we prove various rate estimates for the convergence of the resolvents corresponding to the generators of the approximating semigroups $S^N(t)$ We choose $\kappa \in \mathbf{R}$ such that $0 \in \rho(\mathcal{A} - \kappa\mathcal{T})$. Then Theorem 4.1,b) shows that also $0 \in \rho(\mathcal{A}^N - \kappa\mathcal{T}^N)$ for N sufficiently large, $N \geq N_1$. We put

$$\tilde{\mathcal{A}} = \mathcal{A} - \kappa\mathcal{T} \quad \text{and} \quad \tilde{\mathcal{A}}^N = \mathcal{A}^N - \kappa\mathcal{T}^N, \quad N \geq N_1.$$

Furthermore we introduce the notation

$$\tilde{L}(\varphi) = (A_0 - \kappa I_n)\varphi(0) + \sum_{j=1}^{\ell} A_j\varphi(\theta_j) + \int_{-r}^{0} A(\theta)\varphi(\theta)\,d\theta,$$
$$\tilde{\Delta}^N(\lambda) = \lambda I_n - \tilde{L}(\epsilon_\lambda^N I_n), \quad \tilde{\Delta}(\lambda) = \lambda I_n - \tilde{L}(e^{\lambda\cdot} I_n).$$

The following formula will be of fundamental importance:

$$\begin{aligned}
&P^N(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{A}^N)^{-1}P^N \\
&\quad = (\lambda I - \mathcal{A}^N)^{-1}\Big(P^N(\mathcal{A} - \kappa\mathcal{T}) - (\mathcal{A}^N - \kappa\mathcal{T}^N)P^N\Big)(\lambda I - \mathcal{A})^{-1} \\
&\quad = (\lambda I - \mathcal{A}^N)^{-1}\tilde{\mathcal{A}}^N P^N\Big(\tilde{\mathcal{A}}^{-1}P^N\tilde{\mathcal{A}} - I\Big)(\lambda I - \mathcal{A})^{-1} \\
&\quad = \Big((\lambda I - \mathcal{A}^N)^{-1}(\lambda I - \kappa\mathcal{T}^N) - I\Big)P^N(\tilde{\mathcal{A}}^{-1}P^N\tilde{\mathcal{A}} - I)(\lambda I - \mathcal{A})^{-1}
\end{aligned} \tag{B.1}$$

for $\lambda \in \rho(\mathcal{A})$ and $N \geq N_1$. Here we also have used $\mathcal{A}^N z = \mathcal{A}L^N z$ for $z \in Z^N$. The constants ω, b and a are chosen as in Section 6.

Lemma B.1. *Let $\alpha = 1$ if (H) is satisfied, otherwise $\alpha = 1/2$.*
a) There exist positive constants c and N_1 such that

$$\|P^N(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{A}^N)^{-1}P^N\| \leq c\big(\frac{r}{N}\big)^\alpha |\operatorname{Im}\lambda|^4$$

for all $\lambda \in \Sigma_{a,\omega}$ and $N \geq N_1$.
b) For any compact set $K \subset \rho(\mathcal{A})$ there exist positive constants c and N_1 such that

$$\|P^N(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{A}^N)^{-1}P^N\| \leq c\big(\frac{r}{N}\big)^\alpha$$

for all $\lambda \in K$ and $N \geq N_1$.

Proof: For $\varphi \in H^1$ (i.e., $(\varphi(0), \varphi) \in \operatorname{dom}\mathcal{A}$) we get

$$P^N\tilde{\mathcal{A}}(\varphi(0), \varphi) = P^N(\tilde{L}(\varphi), \dot{\varphi}) = (\tilde{L}(\varphi), \chi^N)$$

with (see Lemma 2.1,a))

$$\chi^N = \frac{N}{r}\sum_{k=1}^{N} E_k^N(\varphi(t_{k-1}^N) - \varphi(t_k^N)).$$

From (1.5) we obtain

$$\tilde{\mathcal{A}}^{-1} P^N \tilde{\mathcal{A}}(\varphi(0), \varphi) = \tilde{\mathcal{A}}^{-1}(\tilde{L}(\varphi), \chi^N) = -(\psi^N(0), \psi^N), \tag{B.2}$$

where (observe $\tilde{\Delta}^N(0) = \tilde{\Delta}(0)$)

$$\psi^N(\theta) = \psi^N(0) + \int_\theta^0 \chi^N(\tau)\, d\tau,$$
$$\psi^N(0) = \tilde{\Delta}(0)^{-1}\Big(\tilde{L}(\varphi) + \tilde{L}\big(\int_\cdot^0 \chi^N(\tau)\, d\tau\big)\Big).$$

It is obvious that $\tilde{\chi}^N(\theta) := \int_\theta^0 \chi^N(\tau)\, d\tau = \sum_{k=1}^N B_k^N(\theta)\tilde{\chi}^N(t_k^N)$ with

$$\tilde{\chi}^N(t_k^N) = \varphi(0) - \varphi(t_k^N), \quad k = 1, \ldots, N.$$

This shows $\tilde{\chi}^N = \varphi(0) - \varphi^N$, where $\varphi^N = \sum_{k=0}^N B_k^N \varphi(t_k^N)$. Therefore

$$\psi^N(0) = \tilde{\Delta}(0)^{-1}\tilde{L}(\varphi - \varphi^N + \varphi(0)) = \tilde{\Delta}(0)^{-1}\tilde{L}(\varphi - \varphi^N) - \varphi(0),$$
$$\psi^N = \tilde{\Delta}(0)^{-1}\tilde{L}(\varphi - \varphi^N) - \varphi(0) + \varphi(0) - \varphi^N = \tilde{\Delta}(0)^{-1}\tilde{L}(\varphi - \varphi^N) - \varphi^N.$$

This together with (B.2) gives

$$(\tilde{\mathcal{A}}^{-1} P^N \tilde{\mathcal{A}} - I)(\varphi(0), \varphi) = -(\tilde{\Delta}(0)^{-1}\tilde{L}(\varphi - \varphi^N), \tilde{\Delta}(0)^{-1}\tilde{L}(\varphi - \varphi^N) + \varphi - \varphi^N). \tag{B.3}$$

If (H) is satisfied we use

$$|\tilde{L}(\varphi - \varphi^N)| = \Big|\int_{-r}^0 A(\theta)(\varphi(\theta) - \varphi^N(\theta))\, d\theta\Big| \le \|A\|_{L^2}\|\varphi - \varphi^N\|_{L^2}, \tag{B.4}$$

otherwise ($\tilde{\gamma} = |A_0 - \kappa I_n| + \sum_{j=1}^\ell |A_j| + \int_{-r}^0 |A(\theta)|\, d\theta$)

$$|\tilde{L}(\varphi - \varphi^N)| \le \tilde{\gamma}\|\varphi - \varphi^N\|_{L^\infty}. \tag{B.5}$$

In case of (B.4) we use ([23; Thm. 2.4])

$$\|\varphi - \varphi^N\|_{L^2} \le \text{const.}\frac{r}{N}\|\varphi\|_{H^1},$$

in case of (B.5) ([23; Exercise (2.10)])

$$\|\varphi - \varphi^N\|_{L^\infty} \le \text{const.}\Big(\frac{r}{N}\Big)^{1/2}\|\varphi\|_{H^1},$$

in order to get from (B.3)

$$\|(\tilde{\mathcal{A}}^{-1} P^N \tilde{\mathcal{A}} - I)(\varphi(0), \varphi)\| \le \text{const.}\Big(\frac{r}{N}\Big)^\alpha\|\varphi\|_{H^1}$$

or, equivalently,

$$\|\tilde{\mathcal{A}}^{-1}P^N\tilde{\mathcal{A}} - I\|_{\mathcal{L}(\mathrm{dom}\,\mathcal{A},Z)} \leq \mathrm{const.}(\frac{r}{N})^\alpha \tag{B.6}$$

for $N = 1, 2, \dots$.

From Theorem 4.1,a) and the estimate $|\lambda| \leq \mathrm{const.}|\mathrm{Im}\,\lambda|$ for $\lambda \in \Sigma_{a,\omega}$ we obtain

$$\left\|\Big((\lambda I - \mathcal{A}^N)^{-1}(\lambda I - \kappa\mathcal{T}^N) - I\Big)P^N\right\|_{\mathcal{L}(Z,Z)} \leq \mathrm{const.}|\mathrm{Im}\,\lambda|^2 \tag{B.7}$$

for all $\lambda \in \Sigma_{a,\omega}$ and $N \geq N_1$. Using $\mathcal{A}(\lambda I - \mathcal{A})^{-1} = \lambda(\lambda I - \mathcal{A})^{-1} - I$ we see that also

$$\|\mathcal{A}(\lambda I - \mathcal{A})^{-1}\| \leq \mathrm{const.}|\mathrm{Im}\,\lambda|^2\|z\|$$

for all $\lambda \in \Sigma_{a,\omega}$ and $z \in Z$. Therefore

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(Z,\mathrm{dom}\,\mathcal{A})} \leq \mathrm{const.}|\mathrm{Im}\,\lambda|^2 \quad \text{for } \lambda \in \Sigma_{a,\omega}. \tag{B.8}$$

Now part a) of the lemma follows from (B.1) and (B.6)–(B.8).

In order to prove part b) we use Theorem 4.1,b) in order to see that the right-hand side in the estimate (B.7) can be replaced by a constant as long as $\lambda \in K$ and $N \geq N_1$. For the estimate (B.8) this is obvious. ∎

LEMMA B.2. *Let $\beta = 2$ if (H) is satisfied, otherwise $\beta = 3/2$.*
a) There exist positive constants c and N_1 such that

$$\|P^N(\lambda I - \mathcal{A})^{-1}z - (\lambda I - \mathcal{A}^N)^{-1}P^N z\| \leq c(\frac{r}{N})^\beta|\mathrm{Im}\,\lambda|^5\|z\|_{\mathrm{dom}\,\mathcal{A}}$$

for all $\lambda \in \Sigma_{a,\omega}$, $z \in \mathrm{dom}\,\mathcal{A}$ and $N \geq N_1$. If $z = (\eta, \varphi)$ with $\varphi \in W^{1,\infty}(-r, 0; \mathbf{R}^n)$ then the right-hand side of the estimate is $c(\frac{r}{N})^2|\mathrm{Im}\,\lambda|^5(|\eta| + \|\varphi\|_{W^{1,\infty}})$.
b) For any compact set $K \subset \rho(\mathcal{A})$ there exist positive constants c and N_1 such that

$$\|P^N(\lambda I - \mathcal{A})^{-1}z - (\lambda I - \mathcal{A}^N)^{-1}P^N z\| \leq c(\frac{r}{N})^\beta\|z\|_{\mathrm{dom}\,\mathcal{A}}$$

for all $\lambda \in K$, $z \in \mathrm{dom}\,\mathcal{A}$ and $N \geq N_1$. If $z = (\eta, \varphi)$ with $\varphi \in W^{1,\infty}(-r, 0; \mathbf{R}^n)$ then the right-hand side of the estimate is $c(\frac{r}{N})^2(|\eta| + \|\varphi\|_{W^{1,\infty}})$.

PROOF: For $z = (\eta, \varphi)$ we put $(\psi(0), \psi) = (\lambda I - \mathcal{A})^{-1}(\eta, \varphi)$. In analogy to (B.3) we get

$$(\tilde{A}^{-1}P^N\tilde{A} - I)(\psi(0), \psi) = -(\tilde{\Delta}(0)^{-1}\tilde{L}(\psi - \psi^N), \tilde{\Delta}(0)^{-1}\tilde{L}(\psi - \psi^N) + \psi - \psi^N), \tag{B.9}$$

where $\psi^N = \sum_{k=0}^N B_k^N\psi(t_k^N)$. Using (4.1) we get (compare the derivation of (4.2))

$$|\Delta(\lambda)^{-1}| \leq \frac{1}{\delta|\lambda|} \quad \text{for } \lambda \in \Sigma_{a,\omega},\ N = 1, 2, \dots .$$

Then

$$|\psi(0)| \leq \frac{1}{\delta|\lambda|}\Big(|\eta| + \gamma \sup_{-r\leq\theta\leq 0}\Big|\int_\theta^0 e^{\lambda(\theta-\tau)}\varphi(\tau)\,d\tau\Big|\Big)$$
$$\leq \text{const.}\|z\| \quad \text{for } \lambda \in \Sigma_{a,\omega}.$$

This together with (4.8) gives

$$\text{(B.10)} \qquad |\psi(\theta)| \leq \text{const.}\|\text{Im}\,\lambda|\,\|z\| \quad \text{for } \lambda \in \Sigma_{a,\omega} \text{ and } -r \leq \theta \leq 0.$$

If φ is absolutely continuous we have by definition of ψ

$$\text{(B.11)} \qquad \dot{\psi} = \lambda\psi - \varphi, \quad \ddot{\psi} = \lambda\dot{\psi} - \dot{\varphi}.$$

Observing $|\lambda| \leq \text{const.}|\text{Im}\,\lambda|$ on $\Sigma_{a,\omega}$ we get from (B.11)

$$\text{(B.12)} \qquad \|\dot{\psi}\|_{L^\infty} \leq \text{const.}|\text{Im}\,\lambda|^2\|z\| \quad \text{for } \lambda \in \Sigma_{a,\omega}$$

and

$$\text{(B.13)} \qquad \|\ddot{\psi}\|_{L^\infty} \leq \text{const.}|\text{Im}\,\lambda|^3\big(|\eta| + \|\varphi\|_{W^{1,\infty}}\big) \quad \text{for } \lambda \in \Sigma_{a,\omega},$$

if $\varphi \in W^{1,\infty}(-r,0;\mathbf{R}^n)$, resp.

$$\text{(B.14)} \qquad \|\ddot{\psi}\|_{L^2} \leq \text{const.}|\text{Im}\,\lambda|^3\big(|\eta| + \|\varphi\|_{W^{1,2}}\big) \quad \text{for } \lambda \in \Sigma_{a,\omega},$$

if $\varphi \in W^{1,2}(-r,0;\mathbf{R}^n)$. (B.14) specifically implies

$$\text{(B.15)} \qquad \|\ddot{\psi}\|_{L^2} \leq \text{const.}|\text{Im}\,\lambda|^3\|z\|_{\text{dom}\,\mathcal{A}} \quad \text{for } \lambda \in \Sigma_{a,\omega},$$

if $z \in \text{dom}\,\mathcal{A}$. Depending whether ψ is in $W^{2,\infty}$ or in H^2 we have the estimates

$$\text{(B.16)} \qquad \|\psi - \psi^N\|_{L^2} \leq \text{const.}\big(\frac{r}{N}\big)^2\|\psi\|_{H^2},$$

$$\text{(B.17)} \qquad \|\psi - \psi^N\|_{L^\infty} \leq \text{const.}\big(\frac{r}{N}\big)^2\|\psi\|_{W^{2,\infty}},$$

$$\text{(B.18)} \qquad \|\psi - \psi^N\|_{L^\infty} \leq \text{const.}\big(\frac{r}{N}\big)^{3/2}\|\psi\|_{H^2}$$

(see [23; Thm 2.5, Thm 2.6 and Exercise (2.10)]). If (H) is satisfied we use (B.16) together with (B.14), otherwise we have to take (B.17) or (B.18) together with (B.13) or (B.14), respectively, in order to get the estimates of part a) from (B.9) and (B.7).

The proof for part b) is analogous but simpler. As in the proof of Lemma B.1 one has to use the fact that the right-hand side of (B.7) can be replaced by a constant for $\lambda \in K$. ∎

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Center for Control Sciences, Brown University, Providence, R.I. 02906
Institute for Mathematics, University of Graz, Elisabethstraße 16, A 8010 Graz (Austria)